The Zakharov and Modified Zakharov Equations and their Applications.

L. SHEMER

Department of Fluid Mechanics, Faculty of Engineering, Tel-Aviv University Tel-Aviv 69978, Israel

M. STIASSNIE

Department of Civil Engineering and Coastal and Marine Engineering Research Institute, Technion - Haifa 32000, Israel

1. – Introduction.

Our understanding of the nonlinear dynamics of water waves has grown substantially in recent years. To a large extent, this progress can be attributed to applications of the so-called Zakharov integral equation, mainly by the group associated with the TRW Fluid Mechanics Department. Most of their findings were summarized in extensive review papers by YUEN and LAKE [1, 2]. Zakharov equation was originally derived by ZAKHAROV [3] for infinitely deep water. ZAKHAROV and KHARITONOV [4] extended the derivation to arbitrary constant water depth, but did not present the interaction coefficients. In his original paper ZAKHAROV [3] has shown that the nonlinear Schrödinger (NLS) equation can be deduced from the Zakharov integral equation when a narrow wave number spectrum is assumed. While the NLS equation served as a main tool in obtaining valuable information about the instability, long-time evolution and recurrence of weakly nonlinear wave trains, it is our opinion that the Zakharov equation is superior to all other existing approximate models as far as class-I interactions of wide spectrum are concerned.

The term «class I» interactions refers to nonlinear interaction processes at the lowest possible order; for surface gravity waves this occurs at third order in the small parameter of the problem ε. Generally speaking, class-I interactions require the coexistence of resonating, or nearly resonating, wave quartets. The time scale of class-I interactions is $\varepsilon^{-2}P$, where P is a typical wave period. The dispersion relation of the surface gravity waves does not enable nonlinear interactions at shorter time scales ($\varepsilon^{-1}P$) which occur in many other physical systems, *e.g.* capillary waves [5-8].

While class-I interactions are basically quartet, or four-wave interactions, the special case where one of the waves, called the «carrier», is taken into account twice so that only three waves are considered has attracted much attention.

These cases, which lead to what is called Benjamin-Feir instability, display many of the features of the more general quartet interactions. Interactions including a smaller number of waves—as two waves each taken into account twice, or one wave counted four times—are also possible, but display a degenerate type of interaction which manifests itself in Stokes-type second-order corrections of the frequency.

Numerical linear stability analysis of exact finite-amplitude Stokes wave, by McLEAN [9, 10], as well as experimental evidence [11, 12] reveal the importance of class-II interactions, which are basically quintet (five-wave) interactions. These much less studied interactions occur at the fourth order of ε and have a typical time scale $\varepsilon^{-3}P$. Nevertheless, for high enough carrier steepness McLean's study, as well as the earlier work of Longuet-Higgins [13], show that class-II instabilities become dominant. Here again, three waves—carrier taken into account three times and two additional «disturbances»—form a nearly resonating quintet and display many interesting features.

The present lecture does not pretend to give a full picture regarding the potential use of the Zakharov integral equation. Instead, we present here the main results of our recent works. The modified Zakharov equation for arbitrary water depth which allows one to analyse also the class II is derived in sect. 2. This derivation, which follows the lines of ref. [2], is based on [14]. The analysis of interactions of a degenerate problem, which includes two nonlinear wave trains, is presented in sect. 3 and is based on the paper [15]. The linear stability of a uniform wave train to both class-I and class-II disturbances is investigated in sect. 4 and is based on ref. [14]. In sect. 5 the long-time evolution of Stokes waves is studied. The three-wave system is considered first. In this case for class-I interactions an analytical solution for the recurrence period of the long-time modulation is obtained, following [16]. The coupled evolution of the five-wave system, which includes the most unstable disturbances of class-I and class-II interactions, is then considered, as reported in [17]. Note that, since the interaction coefficients involved are very cumbersome, we do not present their exact form in the present lecture and address the reader to the appendices of ref. [14, 17]. In the last section we discuss the advantages and disadvantages of the Zakharov equation approach as compared to the alternative nonlinear models, and suggest some possible additional applications.

2. – Derivation of the modified Zakharov equation.

The Laplace equation

$$\nabla^2 \phi = 0 \qquad (-h \leqslant z \leqslant \eta(\boldsymbol{x}, t)) \tag{2.1}$$

describes the irrotational flow of an incompressible inviscid fluid, where ϕ is the velocity potential, η and h are the locations of the free surface and the bottom, respectively, and the vertical coordinate z is pointing upward. The horizontal coordinates are $(x_1, x_2) = \boldsymbol{x}$, and t is the time. The governing equation (1) is subject to the kinematic and the dynamic boundary conditions at the free surface $z = \eta(\boldsymbol{x}, t)$:

$$\eta_t + (\nabla\phi)\cdot(\nabla\eta) - \phi_z = 0\,, \tag{2.2a}$$

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz = 0\,. \tag{2.2b}$$

The boundary condition at the bottom $z = -h$ is

$$\phi_z = 0\,. \tag{2.3}$$

The boundary conditions (2.2) are rewritten in terms of the velocity potential at the free surface, ϕ^s, and the vertical velocity component at the free surface, w^s:

$$\eta_t + (\nabla_x\phi^s)\cdot(\nabla_x\eta) - w^s[1 + (\nabla_x\eta)^2] = 0\,, \tag{2.4a}$$

$$\phi_t^s + g\eta + \frac{1}{2}(\nabla_x\phi^s)^2 - 1/2(w^s)^2[1 + (\nabla_x\eta)^2] = 0\,, \tag{2.4b}$$

where $\nabla_x = (\partial/\partial x_1)\,\hat{\imath} + (\partial/\partial x_2)\,\hat{\jmath}$ is the horizontal operator and g is the acceleration of gravity. The horizontal Fourier transform of eqs. (2.4) yields

$$\begin{aligned}\hat{\eta}_t(\boldsymbol{k}, t) - \frac{1}{2\pi}\iint_{-\infty}^{\infty}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\,\hat{\phi}^s(\boldsymbol{k}_1, t)\,\hat{\eta}(\boldsymbol{k}_2, t)\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 - \hat{w}^s +\\ + \frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\,\hat{w}^s(\boldsymbol{k}_1, t)\,\hat{\eta}(\boldsymbol{k}_2, t)\,\hat{\eta}(\boldsymbol{k}_3, t)\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 = 0\,,\end{aligned} \tag{2.5a}$$

$$\begin{aligned}\hat{\phi}_t^s(\boldsymbol{k}, t) + g\hat{\eta}(\boldsymbol{k}, t) - \frac{1}{4\pi}\iint_{-\infty}^{\infty}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\,\hat{\phi}^s(\boldsymbol{k}_1, t)\,\hat{\phi}^s(\boldsymbol{k}_2, t)\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 -\\ - \frac{1}{4\pi}\iint_{-\infty}^{\infty}\hat{w}^s(\boldsymbol{k}_1, t)\,\hat{w}^s(\boldsymbol{k}_2, t)\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2 +\\ + \frac{1}{16\pi^3}\iiiint_{-\infty}^{\infty}(\boldsymbol{k}_3\cdot\boldsymbol{k}_4)\,\hat{w}^s(\boldsymbol{k}_1, t)\,\hat{w}^s(\boldsymbol{k}_2, t)\,\hat{\eta}(\boldsymbol{k}_3, t)\,\hat{\eta}(\boldsymbol{k}_4, t)\,\cdot\\ \cdot\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4 = 0\,,\end{aligned} \tag{2.5b}$$

where the two-dimensional Fourier transform of a function $f(\boldsymbol{x})$ is given by

$$\hat{f}(\boldsymbol{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\boldsymbol{x})\exp[-i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{x}\,,$$

the Dirac δ-function is defined as

$$\delta(\boldsymbol{k}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{x}$$

and use is made of the convolution and differentiation theorems for the Fourier transform.

The Fourier transform of the Laplace equation (2.1) gives

$$\frac{\partial^2 \hat{\phi}(\boldsymbol{k}, z, t)}{\partial z^2} + \boldsymbol{k}\cdot\boldsymbol{k}\,\hat{\phi}(\boldsymbol{k}, z, t) = 0\,,$$

which together with the boundary condition at the bottom (2.3) yields

$$\hat{\phi}(\boldsymbol{k}, z, t) = \hat{\Phi}(\boldsymbol{k}, t)\cosh(|\boldsymbol{k}|(z+h))\,. \tag{2.6}$$

Opening the brackets in (2.6), substituting $z = \eta(\boldsymbol{x}, t)$ and taking the inverse Fourier transform gives for the velocity potential at the free surface $\phi^s(\boldsymbol{x}, t)$

$$\phi^s(\boldsymbol{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\Phi}(\boldsymbol{k}, t)[\cosh(|\boldsymbol{k}|h)\cosh(|\boldsymbol{k}|\eta(\boldsymbol{x}, t)) + \\ + \sinh(|\boldsymbol{k}|h)\sinh(|\boldsymbol{k}|\eta(\boldsymbol{x}, t))]\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{k}\,. \tag{2.7a}$$

In a similar way, differentiating (2.6) with respect to z, we obtain for the vertical velocity at the free surface

$$w^s(\boldsymbol{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\boldsymbol{k}|\hat{\Phi}(\boldsymbol{k}, t)[\cosh(|\boldsymbol{k}|h)\sinh(|\boldsymbol{k}|\eta(\boldsymbol{x}, t)) + \\ + \sinh(|\boldsymbol{k}|h)\cosh(|\boldsymbol{k}|\eta(\boldsymbol{x}, t))]\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{k}\,. \tag{2.7b}$$

In the sequel, $\hat{\phi}^s$ and $\hat{\eta}$ serve as independent variables. The vertical velocity $\hat{w}^s$ has, therefore, to be expressed as a function of these variables. We make here an additional physical assumption that the wave steepness is small, *i.e.* $|\boldsymbol{k}|\eta = o(1)$. This assumption allows us to expand all functions of $|\boldsymbol{k}|\eta$ in (2.7*a*), (2.7*b*) in the Taylor series up to the order $(|\boldsymbol{k}|\eta)^3$. The surface elevation $\eta(\boldsymbol{x}, t)$ can be expressed in (2.7*a*), (2.7*b*) by its inverse Fourier transform:

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(\boldsymbol{k}, t)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{k}\,.$$

Finally, the Fourier transform of (2.7*a*), (2.7*b*) yields

$$\hat{\phi}^s(\boldsymbol{k}, t) = \hat{\Phi}(\boldsymbol{k}, t)\cosh(|\boldsymbol{k}|h) + \\ + \frac{1}{2\pi}\iint_{-\infty}^{\infty} |\boldsymbol{k}_1|\sinh(|\boldsymbol{k}_1|h)\hat{\Phi}(\boldsymbol{k}_1, t)\,\hat{\eta}(\boldsymbol{k}_2, t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 + \tag{2.8a}$$

$$+\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}\frac{1}{2}|\boldsymbol{k}_1|^2\cosh(|\boldsymbol{k}_1|h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\cdot$$
$$\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3+$$
$$+\frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty}\frac{1}{6}|\boldsymbol{k}_1|^3\sinh(|\boldsymbol{k}_1|h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t)\cdot$$
$$\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,,$$

$$(2.8b)\qquad \hat{w}^{s}(\boldsymbol{k},t)=|\boldsymbol{k}|\hat{\Phi}(\boldsymbol{k},t)\sinh(|\boldsymbol{k}|h)+$$
$$+\frac{1}{2\pi}\iint_{-\infty}^{\infty}|\boldsymbol{k}_1|^2\cosh(|\boldsymbol{k}_1|h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2+$$
$$+\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}\frac{1}{2}|\boldsymbol{k}_1|^3\sinh(|\boldsymbol{k}_1|h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3+$$
$$+\frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty}\frac{1}{6}|\boldsymbol{k}_1|^4\cosh(|\boldsymbol{k}_1|h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t)\cdot$$
$$\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,.$$

Inverting (2.8*a*) iteratively, we obtain the dependence of $\hat{\Phi}(\boldsymbol{k})$ on the independent variable $\hat{\varphi}^{s}$:

$$\hat{\Phi}(\boldsymbol{k})=\frac{\hat{\varphi}^{s}(\boldsymbol{k},t)}{\cosh(|\boldsymbol{k}|h)}-$$
$$-\frac{1}{\cosh(|\boldsymbol{k}|h)}\,\frac{1}{2\pi}\iint_{-\infty}^{\infty}|\boldsymbol{k}_1|\hat{\varphi}^{s}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\mathrm{tgh}(|\boldsymbol{k}_1|h)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2-$$
$$-\frac{1}{\cosh(|\boldsymbol{k}|h)}\,\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}\frac{|\boldsymbol{k}_1|}{4}[2|\boldsymbol{k}_1|-$$
$$-\,\mathrm{tgh}(|\boldsymbol{k}_1|h)\cdot(|\boldsymbol{k}-\boldsymbol{k}_2|\cdot\mathrm{tgh}(|\boldsymbol{k}-\boldsymbol{k}_2|h)+|\boldsymbol{k}-\boldsymbol{k}_3|\cdot\mathrm{tgh}(|\boldsymbol{k}-\boldsymbol{k}_3|h)+$$
$$+\,|\boldsymbol{k}_1+\boldsymbol{k}_2|\cdot\mathrm{tgh}(|\boldsymbol{k}_1+\boldsymbol{k}_2|h)+|\boldsymbol{k}_1+\boldsymbol{k}_3|\cdot\mathrm{tgh}(|\boldsymbol{k}_1+\boldsymbol{k}_3|h)]\,\hat{\varphi}^{s}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\cdot$$
$$\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3-$$
$$-\frac{1}{\cosh(|\boldsymbol{k}|h)}\,\frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty}\left\{\frac{|\boldsymbol{k}_1|^3}{6}\mathrm{tgh}(|\boldsymbol{k}_1|h)-\frac{1}{2}|\boldsymbol{k}_1|\cdot\mathrm{tgh}(|\boldsymbol{k}_1|h)\cdot|\boldsymbol{k}_1+\boldsymbol{k}_2|^2-\right.$$
$$-\frac{1}{2}|\boldsymbol{k}-\boldsymbol{k}_2|\cdot\mathrm{tgh}(|\boldsymbol{k}-\boldsymbol{k}_2|h)\cdot|\boldsymbol{k}_1|\,[|\boldsymbol{k}_1|-\mathrm{tgh}(|\boldsymbol{k}_1|h)]\cdot[|\boldsymbol{k}_1+\boldsymbol{k}_3|\cdot\mathrm{tgh}(|\boldsymbol{k}_1+\boldsymbol{k}_3|h)+$$
$$\left.+\,|\boldsymbol{k}_1+\boldsymbol{k}_4|\cdot\mathrm{tgh}(|\boldsymbol{k}_1+\boldsymbol{k}_4|h)]\right\}\cdot$$
$$\cdot\,\hat{\varphi}^{s}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,.$$

Substituting this result into (2.8b) yields

$$\begin{aligned}(2.9)\qquad \hat{w}^s(\boldsymbol{k},t) &= |\boldsymbol{k}|\,\mathrm{tgh}(|\boldsymbol{k}|h)\,\hat{\phi}^s(\boldsymbol{k},t) - \\ &- \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} |\boldsymbol{k}_1|\,[|\boldsymbol{k}|\,\mathrm{tgh}(|\boldsymbol{k}|h)\,\mathrm{tgh}(|\boldsymbol{k}_1|h) - |\boldsymbol{k}_1|]\,\hat{\phi}^s(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\cdot \\ &\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 - \\ &- \frac{1}{(2\pi)^2}\iiint\limits_{-\infty}^{\infty} S^{(1)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,\hat{\phi}^s(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\cdot \\ &\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 - \\ &- \frac{1}{(2\pi)^3}\iiiint\limits_{-\infty}^{\infty} S^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\,\hat{\phi}^s(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t)\,\cdot \\ &\qquad\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,.\end{aligned}$$

The kernels $S^{(1)}$ and $S^{(2)}$, as well as other kernels which appear throughout the derivation, are given in ref. [14].

The vertical velocity at the free surface w^s, given by (2.9), is now substituted into (2.5a), (2.5b), so that the terms up to the 4th order of magnitude are retained. In this manner equations are obtained which include only $\hat{\phi}^s$ and $\hat{\eta}$ as the variables. We multiply (2.5a) by $[g/2\omega(\boldsymbol{k})]^{1/2}$ and (2.5b) by $i[\omega(\boldsymbol{k})/2g]^{1/2}$, where the wave frequency ω is related to the wave number by the dispersion relation

$$(2.10)\qquad \omega(\boldsymbol{k}) = [g|\boldsymbol{k}|\,\mathrm{tgh}\,(|\boldsymbol{k}|h)]^{1/2}.$$

Combining these equations yields for a new complex variable

$$(2.11)\qquad b(\boldsymbol{k},t) = \left(\frac{g}{2\omega(\boldsymbol{k})}\right)^{1/2}\hat{\eta}(\boldsymbol{k},t) + i\left(\frac{\omega(\boldsymbol{k})}{2g}\right)^{1/2}\hat{\phi}^s(\boldsymbol{k},t)\,;$$

the following equation is obtained:

$$\begin{aligned}(2.12)\quad & b_t(\boldsymbol{k},t) + i\omega(\boldsymbol{k})\,b(\boldsymbol{k},t) + i\iint\limits_{-\infty}^{\infty} V^{(1)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,b(\boldsymbol{k}_1,t)\,b(\boldsymbol{k}_2,t)\,\cdot \\ &\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 + \\ &+ i\iint\limits_{-\infty}^{\infty} V^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,b^*(\boldsymbol{k}_1,t)\,b(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 + \\ &+ i\iint\limits_{-\infty}^{\infty} V^{(3)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,b^*(\boldsymbol{k}_1,t)\,b^*(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 + \\ &+ i\iiint\limits_{-\infty}^{\infty} W^{(1)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,b(\boldsymbol{k}_1,t)\,b(\boldsymbol{k}_2,t)\,b(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 +\end{aligned}$$

$$+i\iiint_{-\infty}^{\infty} W^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\, b^*(\boldsymbol{k}_1,t)\, b(\boldsymbol{k}_2,t)\, b(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3+$$

$$+i\iiint_{-\infty}^{\infty} W^{(3)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\, b^*(\boldsymbol{k}_1,t)\, b^*(\boldsymbol{k}_2,t)\, b(\boldsymbol{k}_3,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3+$$

$$+i\iiint_{-\infty}^{\infty} W^{(4)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\, b^*(\boldsymbol{k}_1,t)\, b^*(\boldsymbol{k}_2,t)\, b^*(\boldsymbol{k}_3,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3+$$

$$+i\iiiint_{-\infty}^{\infty} X^{(1)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\, b(\boldsymbol{k}_1,t)\, b(\boldsymbol{k}_2,t)\, b(\boldsymbol{k}_3,t)\, b(\boldsymbol{k}_4,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4+$$

$$+i\iiiint_{-\infty}^{\infty} X^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\, b^*(\boldsymbol{k}_1,t)\, b(\boldsymbol{k}_2,t)\, b(\boldsymbol{k}_3,t)\, b(\boldsymbol{k}_4,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4+$$

$$+i\iiiint_{-\infty}^{\infty} X^{(3)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\, b^*(\boldsymbol{k}_1,t)\, b^*(\boldsymbol{k}_2,t)\, b(\boldsymbol{k}_3,t)\, b(\boldsymbol{k}_4,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4+$$

$$+i\iiiint_{-\infty}^{\infty} X^{(4)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\, b^*(\boldsymbol{k}_1,t)\, b^*(\boldsymbol{k}_2,t)\, b^*(\boldsymbol{k}_3,t)\, b(\boldsymbol{k}_4,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4+$$

$$+i\iiiint_{-\infty}^{\infty} X^{(5)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\, b^*(\boldsymbol{k}_1,t)\, b^*(\boldsymbol{k}_2,t)\, b^*(\boldsymbol{k}_3,t)\, b^*(\boldsymbol{k}_4,t)\,\cdot$$

$$\cdot\,\delta(\boldsymbol{k}+\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,,$$

where * denotes the complex conjugate.

The relations between $\hat{\eta}$, $\hat{\phi}^s$ and the complex «amplitude spectrum» b are obtained from (2.11) and its complex conjugate:

$$\hat{\eta}(\boldsymbol{k},t)=\sqrt{\frac{\omega(\boldsymbol{k})}{2g}}\,[b(\boldsymbol{k},t)+b^*(-\boldsymbol{k},t)]\,, \tag{2.13a}$$

$$\hat{\phi}^s(\boldsymbol{k},t)=-i\sqrt{\frac{g}{2\omega(\boldsymbol{k})}}\,[b(\boldsymbol{k},t)-b^*(-\boldsymbol{k},t)]\,. \tag{2.13b}$$

Equation (2.12) is an «exact» equation up to the 4th order in the small parameter of the problem, $|\boldsymbol{k}|\eta$. Its complexity, however, prevents one from using it in practice in most cases. We, therefore, have to use a multiple-scale approach assuming that the wave field can be divided in a slowly varying in time component B and small but rapidly bound components B', B'', B''' and that most of the energy in the wave field is contained in B. These assumptions permit us to write

$$(2.14)\quad b(\boldsymbol{k}, t) = \\ = [\varepsilon\widetilde{B}(\boldsymbol{k}, t_2, t_3) + \varepsilon^2 B'(\boldsymbol{k}, t, t_2, t_3) + \varepsilon^3 B''(\boldsymbol{k}, t, t_2, t_3) + \varepsilon^4 B'''(\boldsymbol{k}, t, t_2, t_3)]\exp[-i\omega(\boldsymbol{k})t],$$

where ε is the small parameter of the problem, and the slow time scales are defined by $t_2 = \varepsilon^2 t$, $t_3 = \varepsilon^3 t$. Note that the slow time $t_1 = \varepsilon t$ is omitted from (2.14). This results from the fact that no resonant interactions of three waves, which would occur at the typical time scale t_1, are possible for surface gravity waves as long as capillary effects are neglected. A modification of the derivation which takes capillarity into account is presented in ref. [5].

Substituting $b(\boldsymbol{k}, t)$ from (2.14) into (2.12) and separating the terms according to their order in ε yields the following results.

Order ε is satisfied identically.

Order ε^2 gives the following equation for B':

$$(2.15a)\quad i\frac{\partial B'}{\partial t} = \iint_{-\infty}^{\infty} [V^{(1)}_{0,1,2}\widetilde{B}_1\widetilde{B}_2\delta_{0-1-2}\exp[i(\omega-\omega_1-\omega_2)t] + \\ + V^{(2)}_{0,1,2}\widetilde{B}^*_1\widetilde{B}_2\delta_{0+1-2}\exp[i(\omega+\omega_1-\omega_2)t] + \\ + V^{(3)}_{0,1,2}\widetilde{B}^*_1\widetilde{B}^*_2\delta_{0+1+2}\exp[i(\omega+\omega_1+\omega_2)t]]\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2,$$

where a compact notation was introduced, in which the arguments $\boldsymbol{k}_i$ of all functions are replaced by subscripts i, with the subscript zero assigned to $\boldsymbol{k}$. Integrating (2.15a) with respect to t, while keeping t_2 and t_3 constant, gives

$$(2.15b)\quad B' = -\iint_{-\infty}^{\infty}\left[V^{(1)}_{0,1,2}\widetilde{B}_1\widetilde{B}_2\delta_{0-1-2}\frac{\exp[i(\omega-\omega_1-\omega_2)t]}{\omega-\omega_1-\omega_2} + \right. \\ + V^{(2)}_{0,1,2}\widetilde{B}^*_1\widetilde{B}_2\delta_{0+1-2}\frac{\exp[i(\omega+\omega_1-\omega_2)t]}{\omega+\omega_1-\omega_2} + \\ \left. + V^{(3)}_{0,1,2}\widetilde{B}^*_1\widetilde{B}^*_2\delta_{0+1+2}\frac{\exp[i(\omega+\omega_1+\omega_2)t]}{\omega+\omega_1+\omega_2}\right]d\boldsymbol{k}_1\,d\boldsymbol{k}_2.$$

It was tacitly assumed here that all the exponents in (2.15) differ from zero under the constraints of the corresponding δ-functions. This assumption fails if, instead of the dispersion relation (2.10), the relation $\omega^2 = g|\boldsymbol{k}| + \sigma|\boldsymbol{k}|^3$, where σ is the coefficient of surface tension, is adopted. In this case resonant interactions of three waves are possible and an appropriate modification of (2.14) and (2.15) is required [5]. These extremely short waves are not considered here. The constant of integration in (2.15b) corresponds to the initial phase and has been set to zero.

At order ε^3 of (2.12) the following equation is obtained:

$$(2.16)\quad i\frac{\partial \widetilde{B}}{\partial t_2}+i\frac{\partial B''}{\partial t}=\iiint_{-\infty}^{\infty}\{\widetilde{T}^{(1)}_{0,1,2,3}\widetilde{B}_1\widetilde{B}_2\widetilde{B}_3\,\delta_{0-1-2-3}\exp[i(\omega-\omega_1-\omega_2-\omega_3)t]+$$
$$+\widetilde{T}^{(2)}_{0,1,2,3}\widetilde{B}_1^*\widetilde{B}_2\widetilde{B}_3\,\delta_{0+1-2-3}\exp[i(\omega+\omega_1-\omega_2-\omega_3)t]+$$
$$+\widetilde{T}^{(3)}_{0,1,2,3}\widetilde{B}_1^*\widetilde{B}_2^*\widetilde{B}_3\,\delta_{0+1+2-3}\exp[i(\omega+\omega_1+\omega_2-\omega_3)t]+$$
$$+\widetilde{T}^{(4)}_{0,1,2,3}\widetilde{B}_1^*\widetilde{B}_2^*\widetilde{B}_3^*\,\delta_{0+1+2+3}\exp[i(\omega+\omega_1+\omega_2+\omega_3)t]\}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3\,.$$

Equation (2.16) consists of terms of two types: those that depend on the fast time t and those that depend only on the slow times. This enables to split (2.16) into two separate equations, one representing the dependence on slow time scale:

$$(2.17)\quad i\frac{\partial \widetilde{B}}{\partial t_2}=\iiint_{-\infty}^{\infty}T^{(2)}_{0,1,2,3}\widetilde{B}_1^*\widetilde{B}_2\widetilde{B}_3\,\delta_{0+1-2-3}\exp[i(\omega+\omega_1-\omega_2-\omega_3)t]\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3\,,$$

and the 2nd, giving the behaviour on the fast time t:

$$(2.18a)\quad i\frac{\partial B''}{\partial t}=\iiint_{-\infty}^{\infty}\{\widetilde{T}^{(1)}_{0,1,2,3}\widetilde{B}_1\widetilde{B}_2\widetilde{B}_3\,\delta_{0-1-2-3}\exp[i(\omega-\omega_1-\omega_2-\omega_3)t]+$$
$$+(\widetilde{T}^{(2)}_{0,1,2,3}-T^{(2)}_{0,1,2,3})\widetilde{B}_1^*\widetilde{B}_2\widetilde{B}_3\,\delta_{0+1-2-3}\exp[i(\omega+\omega_1-\omega_2-\omega_3)t]+$$
$$+\widetilde{T}^{(3)}_{0,1,2,3}\widetilde{B}_1^*\widetilde{B}_2^*\widetilde{B}_3\,\delta_{0+1+2-3}\exp[i(\omega+\omega_1+\omega_2-\omega_3)t]+$$
$$+\widetilde{T}^{(4)}_{0,1,2,3}\widetilde{B}_1^*\widetilde{B}_2^*\widetilde{B}_3^*\,\delta_{0+1+2+3}\exp[i(\omega+\omega_1+\omega_2+\omega_3)t]\}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3\,.$$

The term with $T^{(2)}$ obtained here special treatment, since its exponent may become close to zero under the constraints imposed by the δ-function. We, therefore, distinguish between the nearly resonating quartets, defined by

$$(2.19a, b)\qquad \boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3=0\,,\qquad |\omega+\omega_1-\omega_2-\omega_3|\leqslant O(\varepsilon^2)\,,$$

for which (2.17) is the so-called Zakharov equation, used as the mathematical model for the so-called class-I, or quartet, resonant interactions. The kernel $T^{(2)}$ of (2.17) is defined as follows:

$$(2.20)\quad T^{(2)}_{0,1,2,3}=\begin{cases}\widetilde{T}^{(2)}_{0,1,2,3}\,, & \text{for nearly resonating quartets defined by (2.19)}\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

Equation (2.18a) can now be integrated with respect to t, so that B'' can be obtained and used at the next order:

$$(2.18b)\qquad B'' = -\iiint_{-\infty}^{\infty}\Big\{\widetilde{T}^{(1)}_{0,1,2,3}\widetilde{B}_1\widetilde{B}_2\widetilde{B}_3\,\delta_{0-1-2-3}\frac{\exp[i(\omega-\omega_1-\omega_2-\omega_3)t]}{\omega-\omega_1-\omega_2-\omega_3}+$$

$$+(\widetilde{T}^{(2)}_{0,1,2,3}-T^{(2)}_{0,1,2,3})\widetilde{B}^*_1\widetilde{B}_2\widetilde{B}_3\,\delta_{0+1-2-3}\frac{\exp[i(\omega+\omega_1-\omega_2-\omega_3)t]}{\omega+\omega_1-\omega_2-\omega_3}+$$

$$+\widetilde{T}^{(3)}_{0,1,2,3}\widetilde{B}^*_1\widetilde{B}^*_2\widetilde{B}_3\,\delta_{0+1+2-3}\frac{\exp[i(\omega+\omega_1+\omega_2-\omega_3)t]}{\omega+\omega_1+\omega_2-\omega_3}+$$

$$+\widetilde{T}^{(4)}_{0,1,2,3}\widetilde{B}^*_1\widetilde{B}^*_2\widetilde{B}^*_3\,\delta_{0+1+2+3}\frac{\exp[i(\omega+\omega_1+\omega_2+\omega_3)t]}{\omega+\omega_1+\omega_2+\omega_3}\Big\}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3\,.$$

When (2.15b) and (2.18b) are substituted into (2.12), the following equation is obtained at the fourth order in ε:

$$(2.21a)\qquad i\frac{\partial\widetilde{B}}{\partial t_3}+i\frac{\partial B'''}{\partial t}=-i\frac{\partial B'}{\partial t_2}+$$

$$+\int_{-\infty}^{\infty}[\widetilde{U}^{(1)}_{0,1,2,3,4}\widetilde{B}_1\widetilde{B}_2\widetilde{B}_3\widetilde{B}_4\,\delta_{0-1-2-3-4}\exp[i(\omega-\omega_1-\omega_2-\omega_3-\omega_4)t]+$$

$$+\widetilde{U}^{(2)}_{0,1,2,3,4}\widetilde{B}^*_1\widetilde{B}_2\widetilde{B}_3\widetilde{B}_4\,\delta_{0+1-2-3-4}\exp[i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)t]+$$

$$+\widetilde{U}^{(3)}_{0,1,2,3,4}\widetilde{B}^*_1\widetilde{B}^*_2\widetilde{B}_3\widetilde{B}_4\,\delta_{0+1+2-3-4}\exp[i(\omega+\omega_1+\omega_2-\omega_3-\omega_4)t]+$$

$$+\widetilde{U}^{(4)}_{0,1,2,3,4}\widetilde{B}^*_1\widetilde{B}^*_2\widetilde{B}^*_3\widetilde{B}_4\,\delta_{0+1+2+3-4}\exp[i(\omega+\omega_1+\omega_2+\omega_3-\omega_4)t]+$$

$$+\widetilde{U}^{(5)}_{0,1,2,3,4}\widetilde{B}^*_1\widetilde{B}^*_2\widetilde{B}^*_3\widetilde{B}^*_4\,\delta_{0+1+2+3+4}\exp[i(\omega+\omega_1+\omega_2+\omega_3+\omega_4)t]\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3\,d\boldsymbol{k}_4\,.$$

Equation (2.21a), again, contains terms which depend on the fast time t only, which give an equation for $\partial B'''/\partial t$, and is irrelevant as long as only quintet interactions are concerned. Only the 2nd and the 3rd integrands in (2.21a) enable resonant quintets and thus describe the interactions on the slow time scale t_3. These resonant quintets represent the so-called class-II interactions and are defined similarly to (2.19):

$$(2.22a,b)\qquad \boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4=0\,,\qquad |\Omega+\Omega_1-\Omega_2-\Omega_3-\Omega_4|\leqslant O(\varepsilon^3)\,,$$

where Ω_j are the «Stokes corrected» frequencies, defined by

$$(2.22c)\qquad \Omega_j=\omega_j+\varepsilon^2\int_{-\infty}^{\infty}e_{j1}\,T_{j1j1}\,|\widetilde{B}_1|^2\,d\boldsymbol{k}_1\,,$$

where e_{j1} equals 1 for $j=1$ and 2 otherwise. The frequency corrections given by

(2.22c) will be explained in the next section. These corrections become necessary at the order of derivation considered here. Similarly to (2.20), we now define

$$U^{(2)}_{0,1,2,3,4} = \begin{cases} \widetilde{U}^{(2)}_{0,1,2,3,4}, & \text{for nearly resonating quintets}, \\ 0, & \text{otherwise}; \end{cases} \tag{2.23a}$$

$$U^{(3)}_{0,1,2,3,4} = \begin{cases} \widetilde{U}^{(3)}_{0,1,2,3,4}, & \text{for nearly resonating quintets}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.23b}$$

The part of (2.21a) representing slow time modulations is

$$i\frac{\partial \widetilde{B}}{\partial t_3} = \tag{2.24}$$

$$= \iiiint_{-\infty}^{\infty} [U^{(2)}_{0,1,2,3,4}\widetilde{B}_1^*\widetilde{B}_2\widetilde{B}_3\widetilde{B}_4\,\delta_{0+1-2-3-4}\exp[i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)\,t] +$$

$$+ U^{(3)}_{0,1,2,3,4}\widetilde{B}_1^*\widetilde{B}_2^*\widetilde{B}_3\widetilde{B}_4\,\delta_{0+1+2-3-4}\exp[i(\omega+\omega_1+\omega_2-\omega_3-\omega_4)\,t]]\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,.$$

Finally, combining (2.17) and (2.24) into a single equation for $B = \varepsilon\widetilde{B}$,

$$i\frac{\partial B}{\partial t} = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3}B_1^*B_2B_3\,\delta_{0+1-2-3}\exp[i(\omega+\omega_1-\omega_2-\omega_3)\,t]\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 + \tag{2.25}$$

$$+\iiiint_{-\infty}^{\infty} U^{(2)}_{0,1,2,3,4}B_1^*B_2B_3B_4\,\delta_{0+1-2-3-4}\exp[i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)\,t] +$$

$$+\iiiint_{-\infty}^{\infty} U^{(3)}_{0,1,2,3,4}B_1^*B_2^*B_3B_4\,\delta_{0+1+2-3-4}\cdot$$

$$\cdot\exp[i(\omega+\omega_1+\omega_2-\omega_3-\omega_4)\,t]\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4\,.$$

The first line of (2.25) is the original Zakharov equation, and is identical to (2.17). The full equation (2.25) is the modification of the Zakharov equation (2.17) which includes the possibility of higher-order interactions.

Integration of (2.21a) for nonresonating quintets yields, similarly to (2.18b),

$$B''' = \tag{2.21b}$$

$$= -\iiiint_{-\infty}^{\infty}\Bigg[\widetilde{U}^{(1)}_{0,1,2,3,4}\widetilde{B}_1\widetilde{B}_2\widetilde{B}_3\widetilde{B}_4\,\delta_{0-1-2-3-4}\frac{\exp[i(\omega-\omega_1-\omega_2-\omega_3-\omega_4)\,t]}{\omega-\omega_1-\omega_2-\omega_3-\omega_4} +$$

$$+ (\widetilde{U}^{(2)}_{0,1,2,3,4} - U^{(2)}_{0,1,2,3,4})\widetilde{B}_1^*\widetilde{B}_2\widetilde{B}_3\widetilde{B}_4\,\delta_{0+1-2-3-4}\frac{\exp[i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)\,t]}{\omega+\omega_1-\omega_2-\omega_3-\omega_4} +$$

$$+ (\widetilde{U}^{(3)}_{0,1,2,3,4} - U^{(3)}_{0,1,2,3,4}) \widetilde{B}_1^* \widetilde{B}_2^* \widetilde{B}_3 \widetilde{B}_4 \delta_{0+1+2-3-4} \frac{\exp[i(\omega + \omega_1 + \omega_2 - \omega_3 - \omega_4)\, t]}{\omega + \omega_1 + \omega_2 - \omega_3 - \omega_4} +$$

$$+ \widetilde{U}^{(4)}_{0,1,2,3,4} \widetilde{B}_1^* \widetilde{B}_2^* \widetilde{B}_3^* \widetilde{B}_4 \delta_{0+1+2+3-4} \frac{\exp[i(\omega + \omega_1 + \omega_2 + \omega_3 - \omega_4)\, t]}{\omega + \omega_1 + \omega_2 + \omega_3 - \omega_4} +$$

$$+ \widetilde{U}^{(5)}_{0,1,2,3,4} \widetilde{B}_1^* \widetilde{B}_2^* \widetilde{B}_3^* \widetilde{B}_4^* \delta_{0+1+2+3+4} \frac{\exp[i(\omega + \omega_1 + \omega_2 + \omega_3 + \omega_4)\, t]}{\omega + \omega_1 + \omega_2 + \omega_3 + \omega_4} \Bigg] \cdot$$

$$\cdot \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3 \, \mathrm{d}\boldsymbol{k}_4 \, .$$

Equation (2.21*b*) will be employed in sect. 5 in the analysis of the wave field energy in the process of long-time modulation.

3. – Interaction of two wave trains.

The Zakharov equation is most effective when applied to a wave field with a discrete wave number spectrum. In the present section we deal with the simplest possible nontrivial nonlinear wave interactions, *i.e.* the interaction of two wave trains, denoted by 1 and 2. It can be easily seen that only resonating quartets, satisfying (2.19) with the equal sign, can be constructed from two different wave numbers $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. No resonant quintet (class II) interactions, as defined by (2.22), are possible. Equation (2.17) which describes the class-I (quartet) interactions is, therefore, used in the present section. Thus we take

$$B(\boldsymbol{k}, t) = B_1(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}_1) + B_2(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}_2) \, . \tag{3.1}$$

Substituting (3.1) into (2.17) gives

$$i \frac{\mathrm{d}B_1}{\mathrm{d}t} = T^{(2)}_{1,1,1,1} |B_1|^2 B_1 + [T^{(2)}_{1,2,1,2} + T^{(2)}_{1,2,2,1}]\, |B_2|^2 B_1 \tag{3.2a}$$

and

$$i \frac{\mathrm{d}B_2}{\mathrm{d}t} = T^{(2)}_{2,2,2,2} |B_2|^2 B_2 + [T^{(2)}_{2,1,2,1} + T^{(2)}_{2,1,1,2}]\, |B_1|^2 B_2 \, . \tag{3.2b}$$

The kernel $T^{(2)}$ is a real function of its variables and is symmetrical with respect to the last pair of arguments. For strict resonance conditions, *i.e.* when inequality in (2.19) is replaced by the equal sign, $T^{(2)}$ is also symmetrical in its first two arguments. We can, therefore, denote $T_1 = T^{(2)}_{1,1,1,1}$, $T_2 = T^{(2)}_{2,2,2,2}$ and $T_{1,2} = T^{(2)}_{1,2,1,2} = T^{(2)}_{1,2,2,1} = T^{(2)}_{2,1,1,2} = T^{(2)}_{2,1,2,1}$.

The solution of the pair of ordinary differential equations (3.2) is given by

$$B_1(t) = A_1 \exp[-i(T_1 A_1^2 + 2T_{1,2} A_2^2)\, t] \, , \tag{3.3a}$$

$$B_2(t) = A_2 \exp[-i(T_2 A_2^2 + 2T_{1,2} A_1^2)\, t] \, , \tag{3.3b}$$

where A_1 and A_2 are constants. We now substitute (3.3) into (3.1) and the result into (2.13a). Taking the inverse Fourier transform of the latter yields the result which can be presented in the following form:

$$\eta(\boldsymbol{x}, t) = a_1 \cos(\boldsymbol{k}_1 \cdot \boldsymbol{x} - \Omega_1 t) + a_2 \cos(\boldsymbol{k}_2 \cdot \boldsymbol{x} - \Omega_2 t)\,, \tag{3.4}$$

where the wave amplitudes a_1 and a_2 are related to the constants A_1 and A_2 by

$$A_i = 2\pi\,(\omega_i/2|\boldsymbol{k}_i|)^{1/2}\,a_i\,, \qquad i = 1, 2\,, \tag{3.5}$$

and the frequencies of the wave trains are

$$\Omega_1 = \omega_1 + T_1 A_1^2 + 2T_{1,2} A_2^2 \tag{3.6a}$$

and

$$\Omega_2 = \omega_2 + T_2 A_2^2 + 2T_{1,2} A_1^2\,. \tag{3.6b}$$

The «Stokes correction» (2.22) used in the derivation of the modified Zakharov equation is a straightforward generalization of (3.6). The second correction term in (3.6a), $2T_{1,2}A_2^2$, represents the nonlinear influence of the second wave train on the first one. The first term $T_1 A_1^2$ appears even when the amplitude of the second wave train a_2 vanishes and is identical to the well-known Stokes correction to the frequency of the nonlinear wave train. It can be easily shown, indeed [14], that for the deep-water case

$$T^{(2)}_{1,1,1,1} = \frac{|\boldsymbol{k}_1|^3}{4\pi^2}\,.$$

Substituting this and (3.5) into (3.6a) and letting $A_2 = 0$ yields

$$\Omega_1 = \omega_1 \left(1 + \frac{1}{2}|\boldsymbol{k}_1|^2 a_1^2\right)$$

in agreement with the correction to the wave frequency due to nonlinearity [18].

The nonlinear frequency corrections due to the wave train itself and due to the existence of another wave train are of the same order, provided both wave trains have the same order of amplitude. The phase speed of the i-th wave train is given by $c_i = \Omega_i / |\boldsymbol{k}_i|$. The change of the phase speed of the weakly nonlinear wave train 2 due to the presence of wave train 1, Δc_2, is given by

$$\Delta c_2 = c_2 - \frac{\omega_2 + T_2 A_2^2}{|\boldsymbol{k}_2|} = \frac{2T_{1,2}A_1^2}{|\boldsymbol{k}_2|}\,. \tag{3.7}$$

Using (3.5), this becomes

$$\Delta c_2 = \frac{4\pi^2 \omega_1}{|\boldsymbol{k}_1|\,|\boldsymbol{k}_2|} T_{1,2}\, a_1^2 . \tag{3.8}$$

The change in the phase velocity of the 2nd wave train depends, therefore, only on the interaction coefficient $T_{1,2}$ and is independent of the amplitude of the wave train, a_2. For gravity waves in water of infinite depth this result was first obtained in ref. [19]. HOGAN *et al.* [15] have shown that (3.8) is valid also for gravity-capillary waves.

4. – Linear stability of a wave train.

In the previous section it was shown that the only effect of the nonlinear interaction of two wave trains is the effective shift in the frequency and thus a variation in the phase speed. We now make an additional step and consider the next possible simplest problem, which includes nonlinear interaction of three waves. It appears that three waves is the minimum number necessary to enable significant nonlinear interaction, *i.e.* change in wave amplitudes, for both class I and class II. For anything exciting to happen, these waves have to form nearly resonating quartet, see (2.19), for class-I interactions, and nearly resonating quintet (2.22), for class-II interactions. In order to get a quartet or a quintet out of three waves, we will count one of them, $\boldsymbol{k}_0$, which will be called the «carrier», twice or three times, respectively. Two additional waves will be denoted by subscripts $_1$ and $_2$. Thus the relations between the wave numbers are

$$2\boldsymbol{k}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2 \qquad \text{for class-I interactions} \tag{4.1a}$$

and

$$3\boldsymbol{k}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2 \qquad \text{for class-II interactions}. \tag{4.1b}$$

The governing equations for class-I interactions in the three-wave world have the following discretized form:

$$i\frac{\mathrm{d}B_0}{\mathrm{d}t} = (T_{0,0,0,0}|B_0|^2 + 2T_{0,1,0,1}|B_1|^2 + 2T_{0,2,0,2}|B_2|^2)B_0 + 2T_{0,0,1,2}B_0^* B_1 B_2 \exp[i\Omega_\mathrm{I} t], \tag{4.2a}$$

$$i\frac{\mathrm{d}B_1}{\mathrm{d}t} = (2T_{1,0,1,0}|B_0|^2 + T_{1,1,1,1}|B_1|^2 + 2T_{1,2,1,2}|B_2|^2)B_1 + T_{1,2,0,0}B_2^* B_0^2 \exp[-i\Omega_\mathrm{I} t], \tag{4.2b}$$

$$(4.2c)\qquad i\frac{\mathrm{d}B_2}{\mathrm{d}t}=(2T_{2,0,2,0}|B_0|^2+2T_{2,1,2,1}|B_1|^2+T_{2,2,2,2}|B_2|^2)B_2+ \\ +T_{2,1,0,0}B_1^*B_0^2\exp[-i\Omega_\mathrm{I}t],$$

where

$$(4.2d)\qquad \Omega_\mathrm{I}=2\omega_0-\omega_1-\omega_2=O(\varepsilon^2).$$

For the interactions of three waves that do not satisfy (4.1*a*) and satisfy (4.1*b*), the discretized form of (2.25) gives

$$(4.3a)\qquad i\frac{\mathrm{d}B_0}{\mathrm{d}t}=(T_{0,0,0,0}|B_0|^2+2T_{0,1,0,1}|B_1|^2+2T_{0,2,0,2}|B_2|^2)B_0+ \\ +2U^{(3)}_{0,0,0,1,2}B_0^{*\,2}B_1B_2\exp[i\Omega_\mathrm{II}t],$$

$$(4.3b)\qquad i\frac{\mathrm{d}B_1}{\mathrm{d}t}=(2T_{1,0,1,0}|B_0|^2+T_{1,1,1,1}|B_1|^2+2T_{1,2,1,2}|B_2|^2)B_1+ \\ +U^{(2)}_{1,2,0,0,0}B_2^*B_0^3\exp[-i\Omega_\mathrm{II}t],$$

$$(4.3c)\qquad i\frac{\mathrm{d}B_2}{\mathrm{d}t}=(2T_{2,0,2,0}|B_0|^2+2T_{2,1,2,1}|B_1|^2+T_{2,2,2,2}|B_2|^2)B_2+ \\ +U^{(2)}_{2,1,0,0,0}B_1^*B_0^3\exp[-i\Omega_\mathrm{II}t],$$

where

$$(4.3d)\qquad \Omega_\mathrm{II}=3\omega_0-\omega_1-\omega_2=O(\varepsilon^3).$$

Note that superscript ^{2} has been omitted in $T^{(2)}$ for the sake of brevity.

To complete the formulation of the mathematical problem represented by the system of the ODE (4.2) or (4.3), we specify the initial conditions

$$(4.4)\qquad B_0(0)=b_0,\qquad B_1(0)=b_1,\qquad B_2(0)=b_2,$$

where the relation between B_i and the actual wave amplitude a_i is obtained from (3.5):

$$a_i=\frac{1}{\pi}\left(\frac{\omega_i}{2g}\right)^{1/2}|B_i|.$$

For the study of the linear stability of a wave train, we assume that the initial amplitude of the carrier, b_0, is much larger than those of the two other waves, which will be called «disturbances». In the following short-time analysis, only linear terms in the disturbances are retained, so that the carrier wave is unaffected by them and is obtained by solving the linearized equation (4.2a) or (4.3a):

$$B_0 = b_0 \cdot \exp[-iT_{0,0,0,0}\, b_0^2 t],$$

where b_0 is assumed to be real without loss of generality.

4'1. *Class*-I *instabilities*. – We express the wave numbers of the carrier and of the disturbances satisfying (4.1a) in the following form:

$$\boldsymbol{k}_0 = k(1,0), \qquad \boldsymbol{k}_1 = k(1+p, q), \qquad \boldsymbol{k}_2 = k(1-p, -q). \tag{4.5}$$

The linearized version of (4.2b), (4.2c) is

$$i\frac{\mathrm{d}B_1}{\mathrm{d}t} = 2T_{1,0,1,0}\, b_0^2 B_1 + T_{1,2,0,0} B_2^* b_0^2 \exp[-i\widetilde{\Omega}_\mathrm{I} t], \tag{4.6a}$$

$$i\frac{\mathrm{d}B_2}{\mathrm{d}t} = 2T_{2,0,2,0}\, b_0^2 B_2 + T_{2,1,0,0} B_1^* b_0^2 \exp[-i\widetilde{\Omega}_\mathrm{I} t], \tag{4.6b}$$

where $\widetilde{\Omega}_\mathrm{I} = \Omega_\mathrm{I} + 2T_{0,0,0,0}\, b_0^2$.

Looking for solution of (4.6) in the form

$$B_1 = b_1 \cdot \exp[-i(0.5\widetilde{\Omega}_\mathrm{I} + \delta_\mathrm{I})t], \qquad B_2 = b_2 \cdot \exp[-i(0.5\widetilde{\Omega}_\mathrm{I} - \delta_\mathrm{I})t],$$

one can show that δ_I is given by

$$\delta_\mathrm{I} = (T_{1,0,1,0} - T_{2,0,2,0})\, b_0^2 \pm D_\mathrm{I}^{1/2}, \tag{4.7}$$

where the discriminant of the 2nd-order algebraic equation D_I is given by

$$D_\mathrm{I} = [0.5\widetilde{\Omega}_\mathrm{I} - (T_{1,0,1,0} + T_{2,0,2,0})\, b_0^2]^2 - T_{1,2,0,0} T_{2,1,0,0}\, b_0^4. \tag{4.8}$$

Positive values of $D_\mathrm{I}(p,q)$ correspond to stability regions in the (p,q)-plane and *vice versa*. The curves $D_\mathrm{I}(p,q) = 0$ form the stability boundaries, and the values of p and q where D_I attains its minimum define the wave number of the most unstable mode. The value of $\sigma_\mathrm{I} = (-D_\mathrm{I}/gk)^{1/2}$ for most unstable mode represents the maximum growth rate.

4'2. *Class*-II *instabilities*. – For this case (4.5) is replaced by

$$\boldsymbol{k}_0 = k(1,0), \qquad \boldsymbol{k}_1 = k(1+p, q), \qquad \boldsymbol{k}_2 = k(2-p, -q) \tag{4.9}$$

and (4.6) by

$$i\frac{\mathrm{d}B_1}{\mathrm{d}t} = 2T_{1,0,1,0}\, b_0^2 B_1 + U^{(2)}_{1,2,0,0,0} B_2^* b_0^3 \exp[-i\widetilde{\Omega}_{\mathrm{II}}\, t]\,, \tag{4.10a}$$

$$i\frac{\mathrm{d}B_2}{\mathrm{d}t} = 2T_{2,0,2,0}\, b_0^2 B_2 + U^{(2)}_{2,1,0,0,0} B_1^* b_0^3 \exp[-i\widetilde{\Omega}_{\mathrm{II}}\, t]\,, \tag{4.10b}$$

where $\widetilde{\Omega}_{\mathrm{II}} = \Omega_{\mathrm{II}} + 3T_{0,0,0,0}\, b_0^2$. Assuming again that solution has the form of

$$B_1 = b_1 \cdot \exp[-i(0.5\widetilde{\Omega}_{\mathrm{II}} + \delta_{\mathrm{II}})\, t]\,, \qquad B_2 = b_2 \cdot \exp[-i(0.5\widetilde{\Omega}_{\mathrm{II}} - \delta_{\mathrm{II}})\, t]\,,$$

one finds that

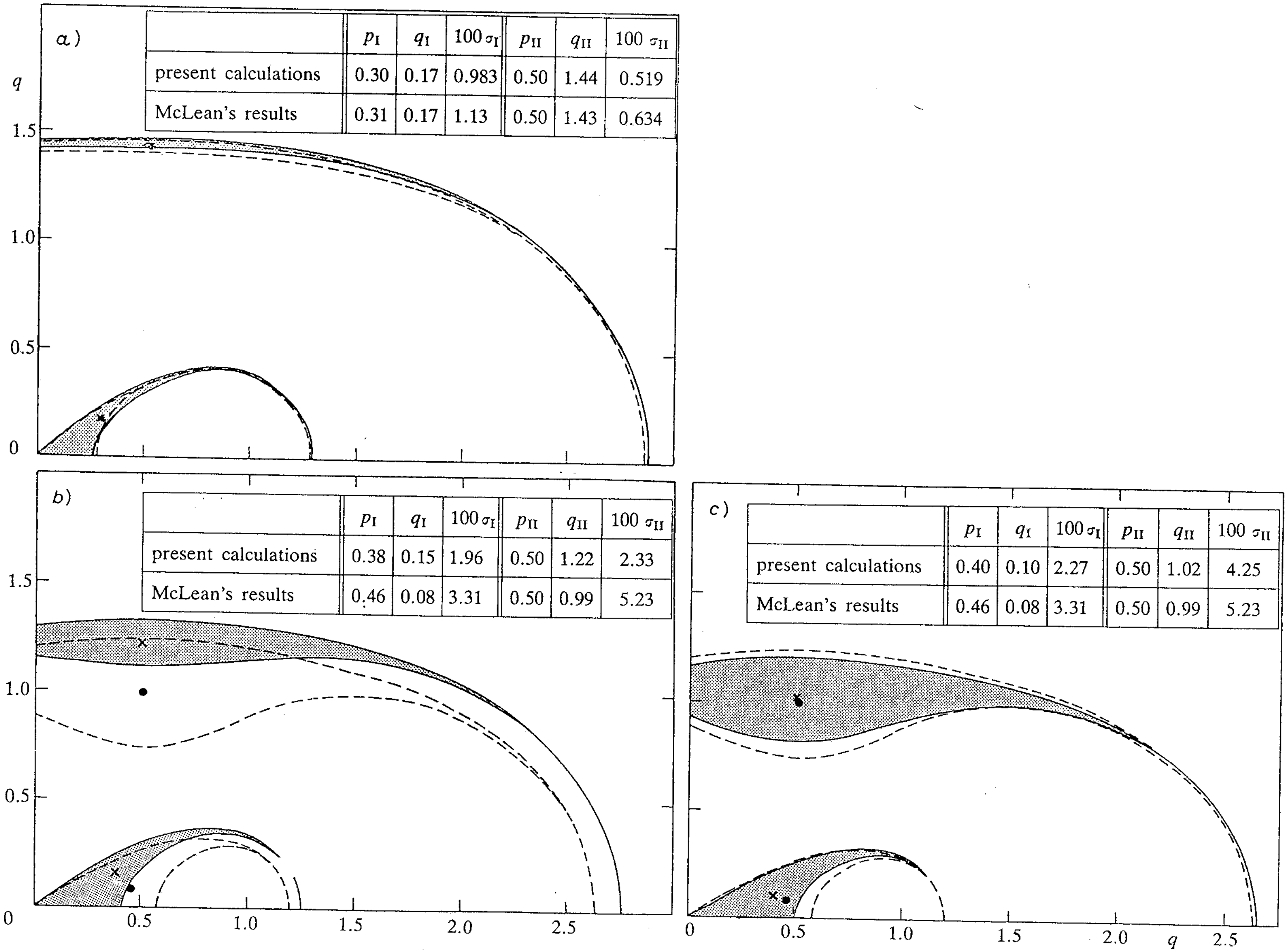
$$\delta_{\mathrm{II}} = (T_{1,0,1,0} - T_{2,0,2,0})\, b_0^2 \pm D_{\mathrm{II}}^{1/2}\,, \tag{4.11}$$

$$D_{\mathrm{II}} = [0.5\widetilde{\Omega}_{\mathrm{II}} - (T_{1,0,1,0} + T_{2,0,2,0})\, b_0^2]^2 - U^{(2)}_{1,2,0,0,0}\, U^{(2)}_{2,1,0,0,0}\, b_0^6\,. \tag{4.12}$$

The stability boundaries, the most unstable mode and the maximum growth rate are obtained from (4.12).

4.3. *Stability regions.* – The instability regions for class-I and class-II interactions for intermediate water depth ($kh = 2$) are shown in fig. 1 as shaded zones. The solid lines present the calculated results and the dashed lines are those of McLean [10]. In fig. 1*a*), $a_0 k = 0.195$, where a_0 is the first-order amplitude of the carrier wave in the Stokes expansion, which corresponds to $(ka)_{\mathrm{M}} = 0.2$ in the calculations of McLean (subscript M stands for McLean). In fig. 1*b*), $a_0 k = 0.326$ (corresponding to $(ka)_{\mathrm{M}} = 0.35$).

The locations of the maximum growth rates $(p_{\mathrm{I}}, q_{\mathrm{I}})$ for class-I, and $(p_{\mathrm{II}}, q_{\mathrm{II}})$ for class-II instabilities, are marked by $\times$ for our results and by dots for McLean's results, and their values, as well as the values of the corresponding maximum growth rates, σ_{I} and σ_{II}, are given in the figures. The overall agreement in fig. 1*a*) is quite satisfactory. The wave amplitude in this case constitutes 47% of the theoretical maximum [20]. For less steep waves, the agreement with McLean's results is even better. For very steep waves (the amplitude in fig. 1*b*) is 82% of the theoretical maximum value), however, the agreement is less impressive. When our results for $a_0 k = 0.41$ are compared with McLean's results for $(ka)_{\mathrm{M}} = 0.35$, the agreement becomes somewhat better. Similar trend in the degree of agreement between the results of McLean [10] and those of ref. [14] model was obtained for several other water depths. Generally speaking, the ref. [14] model gives good quantitative results for amplitudes which do not exceed about one-half of the theoretical maximum. For steeper waves the model still predicts correctly the general qualitative features.



a)

	p_I	q_I	$100\,\sigma_I$	p_{II}	q_{II}	$100\,\sigma_{II}$
present calculations	0.30	0.17	0.983	0.50	1.44	0.519
McLean's results	0.31	0.17	1.13	0.50	1.43	0.634

b)

	p_I	q_I	$100\,\sigma_I$	p_{II}	q_{II}	$100\,\sigma_{II}$
present calculations	0.38	0.15	1.96	0.50	1.22	2.33
McLean's results	0.46	0.08	3.31	0.50	0.99	5.23

c)

	p_I	q_I	$100\,\sigma_I$	p_{II}	q_{II}	$100\,\sigma_{II}$
present calculations	0.40	0.10	2.27	0.50	1.02	4.25
McLean's results	0.46	0.08	3.31	0.50	0.99	5.23

Fig. 1. – Bands of instability for $kh = 2$. The instability boundaries are given by solid lines and the points of maximum growth rate are labelled by $\times$. McLean's results are marked by the dashed lines and by $\bullet$. *a*) $a_0 k = 0.195$, $(ak)_M = 0.2$; *b*) $0.326, 0.35$; *c*) $0.41, 0.35$.

Some of these general features are demonstrated in fig. 2, which is also used to clarify the terminology. Both class-I and class-II instability regions can be regarded as consisting of two domains: a wider band at lower values of p and a usually much narrower band at higher values of p. The first region will be referred to as the main region, and the other as the secondary instability region.

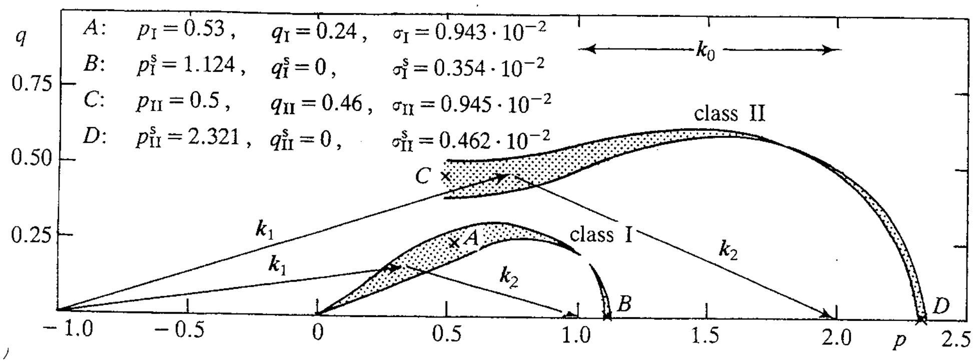


Fig. 2. – Bands of instability for $kh = 0.35$, $a_0 k = 0.04$ and notation.

The difference between class-I and class-II instability regions is that for class I both domains are usually disconnected, while for class II they are bound by a line of infinitesimal thickness. The secondary regions sometimes disappear completely, and, for class I, the instability region in these cases terminates at some $q > 0$. This is similar to the pattern obtained at infinite water depth [21].

The disconnection between the main and the secondary regions for class I, as well as the disappearance of the secondary region in some cases of class-II interactions, were not observed by McLean [9, 10], who solved numerically the full inviscid equations, and, therefore, can possibly be a result of the order of the present perturbation expansion.

Figure 2 shows the three wave number vectors $\boldsymbol{k}_0$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$, which form the wave field, as well as the locations of four points of local maximum growth rates:

A, class-I point (p_I, q_I) with local maximum growth rate σ_I;

B, the secondary class-I point (p_I^s, q_I^s) with local maximum growth rate σ_I^s;

C, class-II point (p_{II}, q_{II}) with local maximum growth rate σ_{II};

D, the secondary class-II point (p_{II}^s, q_{II}^s) with local maximum growth rate σ_{II}^s.

For the particular case presented in fig. 2 ($kh = 0.35$, $a_0 k = 0.04$) $\sigma_{II} > \sigma_I > \sigma_{II}^s > \sigma_I^s$. These inequalities are by no means general, as will be shown in the sequel. In most cases, however, $\sigma_{II} > \sigma_{II}^s$.

Figure 3 is the summary of the results for class-I instabilities. Figures 3*a*), *b*) give the values of p_I and q_I, respectively, as functions of the water depth kh (the range covered is $0.35 < kh < \infty$), and the wave steepness ε^2 as defined by Cokelet [20] and denoted here ε_C^2 (the range $0 \leq \varepsilon_C^2 < 0.7$ is covered). Figure 3*c*) is a plot of the maximum growth rate $\sigma_I^M = \max(\sigma_I, \sigma_I^s)$. Note that for the region

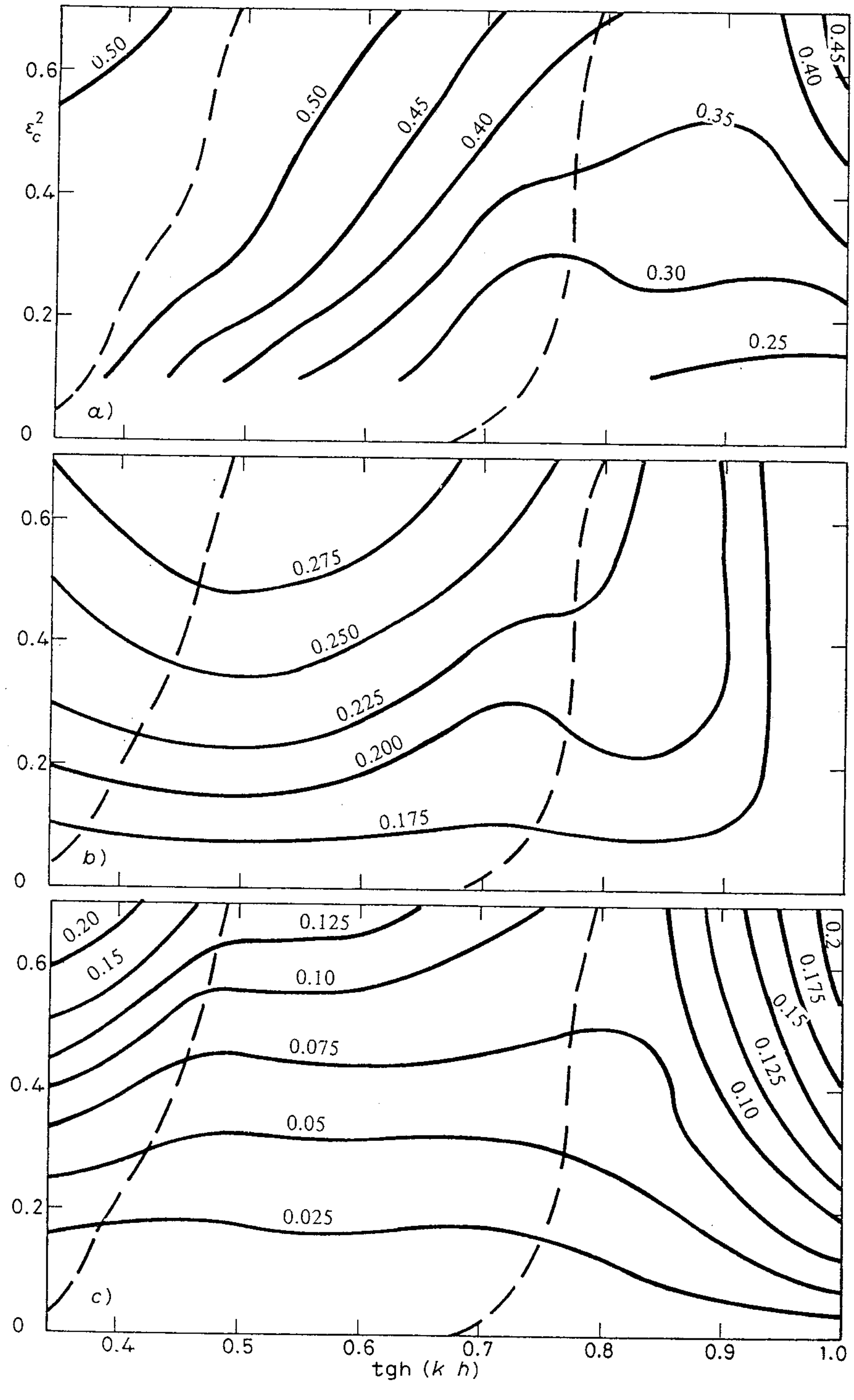


Fig. 3. – Summary of results for class-I instabilities: *a*) isolines of p_{I}, *b*) isolines of q_{I}, *c*) isolines of $10\sigma_{\mathrm{I}}^{\mathrm{M}}$, the maximum growth rate.

confined by the broken lines $\sigma_{\rm I}^{\rm s} > \sigma_{\rm I}$ (sometimes by a factor of three), whereas the opposite is true in the outside region. For the cases where $\sigma_{\rm I}^{\rm s} > \sigma_{\rm I}$, $p_{\rm I}^{\rm s}$ is in the range $1.05 \div 1.30$, while $q_{\rm I}^{\rm s} = 0$, which implies that the most unstable mode is two-dimensional. All isolines in fig. 3 were drawn using interpolation and are based on about forty computed data points, almost equally distributed over the figure domain.

Similarly obtained results for class-II interactions are presented in fig. 4.

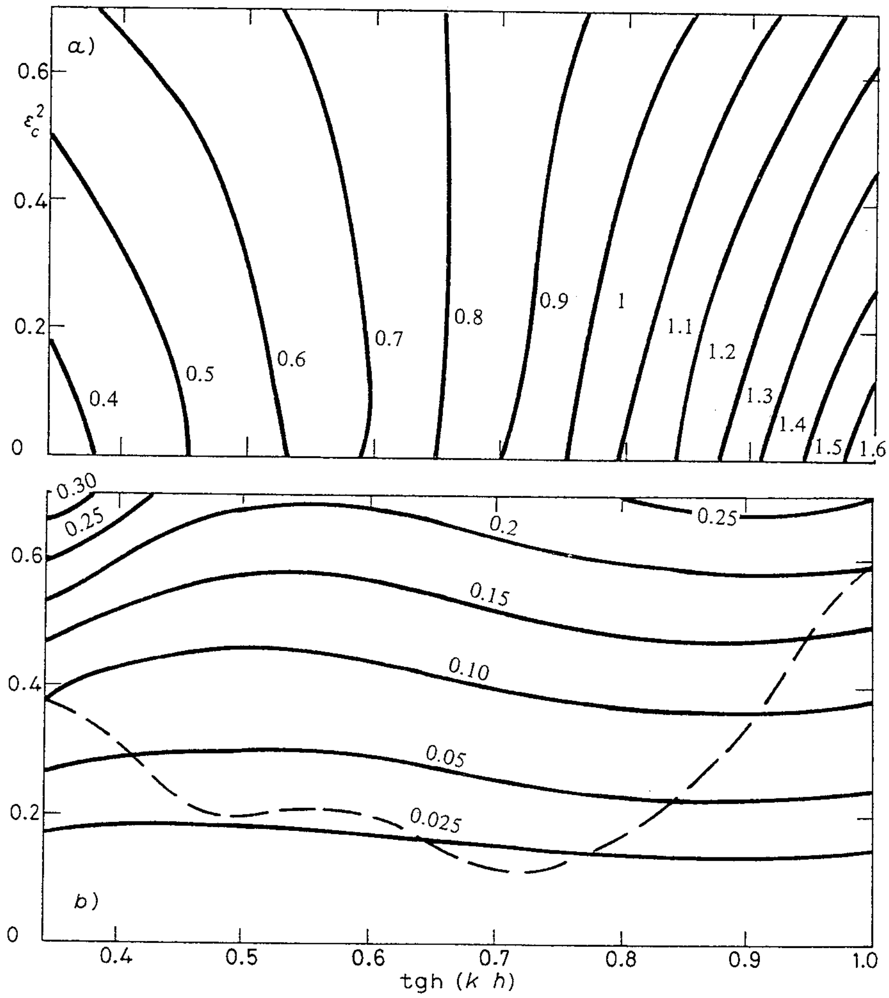


Fig. 4. – Summary of results for class-II instabilities: *a*) isolines of $q_{\rm II}$, *b*) isolines of $10\sigma_{\rm II}$.

Note that in this case $p_{\rm II}$ is always 0.5. Figure 4*a*) gives the values of $q_{\rm II}$, while $\sigma_{\rm II}$ is shown in fig. 4*b*). For the domain above the dashed line in fig. 4*b*), $\sigma_{\rm II} > \sigma_{\rm I}^{\rm M}$, which indicates that for steep waves class-II instabilities may become dominant.

5. – Long-time evolution of Stokes waves.

Wave flume experiments on nonlinear wave trains [22] have shown that the unstable modulations grew to a certain maximum value and then subsided, reaching nearly the undisturbed state again. This cyclic modulation-demodulation phenomenon associated with instabilities of nonlinear systems is known as the Fermi-Pasta-Ulam recurrence. YUEN and FERGUSON [23, 24] have obtained such a periodic recurrence by solving numerically the nonlinear Schrödinger equation. YUEN and LAKE [2] used a numerical solution of the Zakharov equation to show that the evolution may be either recurring (cyclic), or chaotic, depending on the choice of the modes included in the calculations. STIASSNIE and KROSZYNSKI [25] used the nonlinear Schrödinger equation to study analytically the evolution of a simpler, three-wave system, composed of a carrier and two initially small «sideband» disturbances. Their recurrence period (given by a simple formula) is in good agreement with the numerical results.

Experiments by SU [11] and SU *et al.* [12] have shown that an initial state of a steep two-dimensional wave train evolved into series of three-dimensional crescentic spilling breakers, and was followed by a transition to a two-dimensional wave train. One can speculate that the growth of the crescentic waves and their disappearance are one cycle of a recurring phenomenon. The three-dimensional character of these disturbances suggests that they originate from the class-II instabilities. The modified Zakharov equation can serve as a tool for the analytical study of such recurrence phenomena.

In later investigations SU and GREEN [26, 27] suggested the following interpretation of their experimental results: under the initial action of class-I instability a wave train with moderately high steepness ($a_0 k > 0.12$) may undergo a considerable modulation in its envelope; subsequently, a few of the waves in the middle of the maximum modulation will have local wave steepness high enough to trigger class-II instability. They added that for high enough initial steepness ($a_0 k > 0.15$) these locally steep waves lead to three-dimensional wave breaking.

In the present section we study first the long-time evolution of a three-wave system in deep water, which is composed of the carrier and of two most unstable, initially small disturbances, separately for class-I and class-II instabilities. At the next stage, coupled evolution of class I and class II is considered. In this case, a five-wave system, consisting of the carrier and the two most unstable disturbances of each class, is analysed. The total energy of the wave field is also calculated. In any nondissipative system it should remain constant. This can serve as a check of the mathematical model.

5.1. *Evolution of a three-wave system.* – We consider a three-wave system consisting of waves with the wave numbers given by (4.5) for class-I and by (4.9) for class-II interactions. The initial amplitudes and phases are chosen as follows:

$$a_0(0) = \tilde{a}_0, \qquad a_1(0) = a_2(0) = \mu\tilde{a}_0, \qquad \mu = o(1), \tag{5.1a}$$

$$\theta_0(0) = 0, \qquad \theta_1(0) = \theta_{\mathrm{I}}, \qquad \theta_2(0) = \theta_{\mathrm{II}}. \tag{5.1b}$$

The «Stokes-corrected» frequencies $\Omega_i\,(i=0,1,2)$ are given by

$$\Omega_i = \omega_i + T^{(2)}_{i,i,i,i}|B_i|^2 + 2\sum_{i\neq j} T^{(2)}_{i,j,i,j}|B_j|^2\,, \tag{5.2}$$

which follows from (2.22c). The variables B_i are related to the quantities defined by (5.1) and (5.2) by

$$B_i(t) = \pi\left(\frac{2g}{\omega_i}\right)^{1/2} a_i \exp\left[i\left(\int_0^t (\omega_i - \Omega_i)\,\mathrm{d}t + \theta_i\right)\right]. \tag{5.3}$$

The governing equations for B_i are the discretized form of the Zakharov equation (2.17) for class-I instabilities and the modified Zakharov equation (2.25) for class-II instabilities, given by

$$i\frac{\mathrm{d}B_0}{\mathrm{d}t} = (\Omega_0 - \omega_0)B_0 + 2K_0^{(J)}B_0^{*J}B_1B_2\exp[i\Omega_J t]\,, \tag{5.4$_0$}$$

$$i\frac{\mathrm{d}B_1}{\mathrm{d}t} = (\Omega_1 - \omega_1)B_1 + K_1^{(J)}B_2^{*}B_0^{J+1}\exp[-i\Omega_J t]\,, \tag{5.4$_1$}$$

$$i\frac{\mathrm{d}B_2}{\mathrm{d}t} = (\Omega_2 - \omega_2)B_2 + K_2^{(J)}B_1^{*}B_0^{J+1}\exp[-i\Omega_J t]\,, \tag{5.4$_2$}$$

where for class-I instabilities $J=1$ and for class II, $J=2$, Ω_J is defined by (4.2d) or (4.3d), respectively, and the kernels $K_i^{(J)}$ are given by

$$K_0^{(1)} = T^{(2)}_{0,0,1,2}\,,\qquad K_1^{(1)} = T^{(2)}_{1,2,0,0}\,,\qquad K_2^{(1)} = T^{(2)}_{2,1,0,0}\,, \tag{5.5a}$$

$$K_0^{(2)} = \frac{1}{2}\left(U^{(3)}_{0,0,0,1,2} + U^{(3)}_{0,0,0,2,1}\right),\qquad K_1^{(2)} = U^{(2)}_{1,2,0,0,0}\,,\qquad K_2^{(2)} = U^{(2)}_{2,1,0,0,0}\,. \tag{5.5b}$$

From (5.3) and (5.1)

$$B_0(0) = \pi(2g/\omega_0)^{1/2}a_0\,, \tag{5.6a}$$

$$B_1(0) = \pi(2g/\omega_1)^{1/2}a_1\exp[i\theta_{\mathrm{I}}]\,, \tag{5.6b}$$

$$B_2(0) = \pi(2g/\omega_2)^{1/2}a_2\exp[i\theta_{\mathrm{II}}]\,. \tag{5.6c}$$

Applying the operation $-i\cdot B_j^*\cdot$ eq. (5.4j) $+\, i\cdot B_j\cdot$ eq. (5.4j)* on the system (5.4), $j=0,1,2$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_0|^2 = 4K_0^{(J)}\,\mathrm{Im}\left\{(B_0^*)^{J+1}B_1B_2\exp[i\Omega_J t]\right\} = 4K_0^{(J)}\,\mathrm{Im}\,(H)\,, \tag{5.7a}$$

(5.7*b*) $$\frac{d}{dt}|B_1|^2 = -2K_1^{(J)}\,\mathrm{Im}\,\{(B_0^*)^{J+1}B_1B_2\exp[i\Omega_J t]\} = -2K_1^{(J)}\,\mathrm{Im}\,(H)\,,$$

(5.7*c*) $$\frac{d}{dt}|B_2|^2 = -2K_2^{(J)}\,\mathrm{Im}\,\{(B_0^*)^{J+1}B_1B_2\exp[i\Omega_J t]\} = -2K_2^{(J)}\,\mathrm{Im}\,(H)\,,$$

where the complex function

(5.7*d*) $$H(t) = (B_0^*)^{J+1}B_1B_2\exp[i\Omega_J t]\,.$$

A new real function Z is now defined, so that

(5.8) $$\frac{dZ}{dt} = \mathrm{Im}\,(H(t)) = \mathrm{Im}\,\{(B_0^*)^{J+1}B_1B_2\exp[i\Omega_J t]\}\,.$$

Substitution of (5.8) into (5.7) and integration yields

(5.9*a*) $$|B_0(t)|^2 = 4K_0^{(J)}Z + B_0^2(0)\,,$$

(5.9*b*) $$|B_1(t)|^2 = -2K_1^{(J)}Z + B_1^2(0)\,,$$

(5.9*c*) $$|B_2(t)|^2 = -2K_2^{(J)}Z + B_2^2(0)\,.$$

Using (5.4) it can be shown that

(5.10) $$\frac{d}{dt}\mathrm{Re}\,(H) = -[(J+1)\Omega_0 - \Omega_1 - \Omega_2]\frac{dZ}{dt}\,,$$

which after integration gives

(5.11) $$\mathrm{Re}\,(H(t)) = \mathrm{Re}\,(H(0)) - \int_0^Z [(J+1)\Omega_0 - \Omega_1 - \Omega_2]\,dZ\,.$$

Calculating the square absolute value of $H(t)$ from (5.8) and (5.11) and rearranging the terms yields

(5.12) $$\left(\frac{dZ}{dt}\right)^2 = |H(t)|^2 - \left\{\mathrm{Re}\,(H(0)) - \int_0^Z [(J+1)\Omega_0 - \Omega_1 - \Omega_2]\,dZ\right\}^2 =$$
$$= |B_0(t)|^{2(J+1)}|B_1(t)|^2|B_2(t)|^2 -$$
$$- \left\{\mathrm{Re}\,[(B_0^*(0))^{J+1}B_1(0)B_2(0)] - \int_0^Z [(J+1)\Omega_0 - \Omega_1 - \Omega_2]\,dZ\right\}^2.$$

The r.h.s. of (5.12), after substitution of (5.2) and (5.9), is a known polynomial of Z of order $J+3$ denoted by $P_{J+3}(Z)$.

The solution of (5.12) is

$$t = \int_0^Z \frac{dZ}{\sqrt{P_{J+3}(Z)}}, \tag{5.13}$$

where Z is allowed to vary between two neighbouring roots of the polynomial: $Z = Z_L$ and $Z = Z_R$, where $Z_L < 0$ and $Z_R > 0$.

From (5.13) it is clear that Z is periodic in time, and that the recurrence period T is given by

$$T = 2\int_{Z_L}^{Z_R} \frac{dZ}{\sqrt{P_{J+3}(Z)}}. \tag{5.14}$$

For class-I interactions

$$P_4(Z) = \sum_{i=0}^{4} \alpha_i Z^{(4-i)}. \tag{5.15}$$

When $\alpha_0 > 0$, and there are four roots of the equation $P_4(Z) = 0$, which can be arranged in decreasing order so that $Z_4 > Z_3 > 0 > Z_2 > Z_1$, we obtain that $Z_2 = Z_L$ and $Z_3 = Z_R$. Equation (5.12) in this case has an explicit solution

$$Z = \frac{Z_4(Z_3 - Z_2)\,\mathrm{sn}^2(u, \kappa) - Z_3(Z_4 - Z_2)}{(Z_3 - Z_2)\,\mathrm{sn}^2(u, \kappa) - (Z_4 - Z_2)}, \tag{5.16}$$

where $\mathrm{sn}(u, \kappa)$ is the Jacobian elliptic function of argument u and modulus κ:

$$u = \mathrm{sn}^{-1}(\beta, \kappa) - \alpha_0^{1/2}\, t/\gamma, \tag{5.17a}$$

$$\beta = \sqrt{(Z_4 - Z_2)Z_3}/\sqrt{(Z_3 - Z_2)Z_4}, \tag{5.17b}$$

$$\gamma = 2/\sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}, \tag{5.17c}$$

$$\kappa = \sqrt{(Z_3 - Z_2)(Z_4 - Z_1)}/\sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}. \tag{5.17d}$$

The recurrence period for this case is given by

$$T = \frac{2}{\alpha_0^{1/2}} K(\kappa), \tag{5.18}$$

where $K(\kappa)$ is a complete elliptic integral. Expressions similar to (5.17) and (5.18) exist also for $\alpha_0 < 0$. For class II the polynomial $P_5(z)$ is of order five, and thus we cannot express the solution in terms of tabulated functions. Equations (5.13) and (5.14) are, therefore, integrated numerically. Once Z is found, (5.9) is used to obtain $|B_j(t)|$, (5.3) and (5.2) to obtain a_j.

5·2. *Evolution of a five-wave system.* – We now consider a five-wave system, consisting of a carrier and two couples (1, 2) and (3, 4) which represent the most unstable disturbances of class I and class II, respectively. These most unstable disturbances are given by (4.5) and (4.9):

$$\boldsymbol{k}_0 = k(1, 0)\,, \quad \boldsymbol{k}_1 = k(1+p_{\mathrm{I}}, 0)\,, \quad \boldsymbol{k}_2 = k(1-p_{\mathrm{I}}, 0)\,,$$
$$\boldsymbol{k}_3 = k(1.5, q_{\mathrm{II}})\,, \quad \boldsymbol{k}_4 = k(1.5, -q_{\mathrm{II}})\,.$$

The numerical values of p_{I} and q_{II} were obtained in the linear stability analysis in sect. 3 and presented in fig. 3*a*) and 4*a*). The initial amplitudes and phase shifts of these waves are chosen similarly to (5.1):

$$\text{(5.19}a\text{)} \qquad \begin{cases} a_0(0) = \tilde{a}_0\,, \quad a_1(0) = a_2(0) = \mu_{\mathrm{I}}\tilde{a}_0\,, \quad \mu_{\mathrm{I}} = o(1)\,, \\ a_3(0) = a_4(0) = \mu_{\mathrm{II}}\tilde{a}_0\,, \quad \mu_{\mathrm{II}} = o(1); \end{cases}$$

$$\text{(5.19}b\text{)} \qquad \theta_0(0) = 0\,, \quad \theta_1(0) = \theta_2(0) = \theta_{\mathrm{I}}\,, \quad \theta_3(0) = \theta_4(0) = \theta_{\mathrm{II}}\,.$$

The variables B_i satisfy the following discretized version of (2.25):

$$\text{(5.20}a\text{)} \qquad i\frac{\mathrm{d}B_0}{\mathrm{d}t} = (\Omega_0 - \omega_0)B_0 + 2T^{(2)}_{0,0,1,2}B_0^* B_1 B_2 \exp[i\Omega_{\mathrm{I}} t] + $$
$$+ 2U^{(3)}_{0,0,0,3,4}(B_0^*)^2 B_3 B_4 \exp[i\Omega_{\mathrm{II}} t] +$$
$$+ 2(U^{(3)}_{0,1,2,3,4} + U^{(3)}_{0,2,1,3,4})B_1^* B_2^* B_3 B_4 \exp[i(\Omega_{\mathrm{II}} - \Omega_{\mathrm{I}})t]\,,$$

$$\text{(5.20}b\text{)} \qquad i\frac{\mathrm{d}B_1}{\mathrm{d}t} = (\Omega_1 - \omega_1)B_1 + T^{(2)}_{1,2,0,0}B_2^* B_0^2 \exp[-i\Omega_{\mathrm{I}} t] +$$
$$+ 2(U^{(3)}_{1,2,0,3,4} + U^{(3)}_{1,0,2,3,4})B_0^* B_2^* B_3 B_4 \exp[i(\Omega_{\mathrm{II}} - \Omega_{\mathrm{I}})t]\,,$$

$$\text{(5.20}c\text{)} \qquad i\frac{\mathrm{d}B_2}{\mathrm{d}t} = (\Omega_2 - \omega_2)B_2 + T^{(2)}_{2,1,0,0}B_1^* B_0^2 \exp[-i\Omega_{\mathrm{I}} t] +$$
$$+ 2(U^{(3)}_{2,0,1,3,4} + U^{(3)}_{2,1,0,3,4})B_0^* B_1^* B_3 B_4 \exp[i(\Omega_{\mathrm{II}} - \Omega_{\mathrm{I}})t]\,,$$

$$\text{(5.20}d\text{)} \qquad i\frac{\mathrm{d}B_3}{\mathrm{d}t} = (\Omega_3 - \omega_3)B_3 + U^{(2)}_{3,4,0,0,0}B_4^* B_0^3 \exp[-i\Omega_{\mathrm{II}} t] +$$
$$+(U^{(2)}_{3,4,0,1,2} + U^{(2)}_{3,4,0,2,1} + U^{(2)}_{3,4,1,0,2} + U^{(2)}_{3,4,1,2,0} +$$
$$+ U^{(2)}_{3,4,2,0,1} + U^{(2)}_{3,4,2,1,0})B_4^* B_0 B_1 B_2 \exp[i(\Omega_{\mathrm{I}} - \Omega_{\mathrm{II}})t]\,,$$

$$\text{(5.20}e\text{)} \qquad i\frac{\mathrm{d}B_4}{\mathrm{d}t} = (\Omega_4 - \omega_4)B_4 + U^{(2)}_{4,3,0,0,0}B_3^* B_0^3 \exp[-i\Omega_{\mathrm{II}} t] +$$
$$+(U^{(2)}_{4,3,0,1,2} + U^{(2)}_{4,3,0,2,1} + U^{(2)}_{4,3,1,0,2} + U^{(2)}_{4,3,1,2,0} +$$
$$+ U^{(2)}_{4,3,2,0,1} + U^{(2)}_{4,3,2,1,0})B_3^* B_0 B_1 B_2 \exp[i(\Omega_{\mathrm{I}} - \Omega_{\mathrm{II}})t]\,,$$

where Ω_{I} and Ω_{II} are given by (4.2*d*) and (4.3*d*), respectively. From (5.19) and (5.3)

$$B_0(0) = \pi \left(\frac{2g}{\omega_0}\right)^{1/2} \tilde{a}_0 , \tag{5.21a}$$

$$B_1(0) = \pi \left(\frac{2g}{\omega_1}\right)^{1/2} \mu_{\mathrm{I}}\, \tilde{a}_0 \exp[i\theta_{\mathrm{I}}] , \tag{5.21b}$$

$$B_2(0) = \pi \left(\frac{2g}{\omega_2}\right)^{1/2} \mu_{\mathrm{I}}\, \tilde{a}_0 \exp[i\theta_{\mathrm{I}}] , \tag{5.21c}$$

$$B_3(0) = \pi \left(\frac{2g}{\omega_3}\right)^{1/2} \mu_{\mathrm{II}}\, \tilde{a}_0 \exp[i\theta_{\mathrm{II}}] , \tag{5.21d}$$

$$B_4(0) = \pi \left(\frac{2g}{\omega_4}\right)^{1/2} \mu_{\mathrm{II}}\, \tilde{a}_0 \exp[i\theta_{\mathrm{II}}] . \tag{5.21e}$$

The system of 5 nonlinear complex ODEs (5.20) together with the initial values (5.21) forms the evolution problem to be studied. In subsect. 4·1 we studied the degenerate form of the problem, with either B_1 and B_2, or B_3 and B_4 being identically zero. In the more general case of 5 equations (5.20) we cannot reduce the system to a single real equation the solution of which can be presented for class-I interactions in terms of Jacobian elliptic functions. The problem was, therefore, solved numerically using the Gil form of the Runge-Kutta method [28]. The numerical scheme was checked by comparing the results obtained for different integration steps. All numerical results proved to be accurate to five significant digits.

5·3. *Energy balance.* – Another approach for checking the mathematical model and the numerical results is to calculate the total energy of the wave field. Note that this energy approach would still hold when more than 5 free components are included.

The exact equations of motion for water waves (2.1) and (2.2) form a Hamiltonian system as shown by ZAKHAROV [3], MILES [29] and MILDER [30], and the total energy of the entire wave field is conserved. The average energy density, taken over the (x_1, x_2)-plane, is given by

$$h = \lim_{L\to\infty} \frac{1}{(2L)^2} \int_{-L}^{L} \int_{-L}^{L} \frac{1}{2}\left(g\eta^2 + \phi^s \frac{\partial\eta}{\partial t}\right) \mathrm{d}x_1\, \mathrm{d}x_2 . \tag{5.22}$$

Any exact solution of (2.1) and (2.2) should give $h = \text{const}$, for all t (as long as breaking and dissipation effects are not considered). When a truncated version of the governing equations is used, one can expect (5.22) to yield $h(t)$ which is only approximately constant. Note that the assumption of five free waves leads to a large number of bound waves of higher order, see (2.14). The number of waves which contribute to the total energy of the wave field considered results from the structure of (2.15*b*), (2.18*b*) and (2.21*b*), and is equal to $5 + 3\cdot 5^2 + 4\cdot 5^3 + 5\cdot 5^4 = 3705$. Equations (2.13) can thus be rewritten in the following form:

$$\eta = \frac{1}{2\pi}\sum_{n=0}^{3704}\left(\frac{\omega_n}{2g}\right)^{1/2}[\widetilde{B}_n \exp[i(\boldsymbol{k}_n\cdot\boldsymbol{x} + \chi_n t)] + *]\,, \tag{5.23a}$$

$$\phi^s = -\frac{i}{2\pi}\sum_{n=0}^{3704}\left(\frac{2g}{\omega_n}\right)^{1/2}[\widetilde{B}_n \exp[i(\boldsymbol{k}_n\cdot\boldsymbol{x} + \chi_n t)] - *]\,, \tag{5.23b}$$

where, for $0 \leqslant n \leqslant 4$, $\widetilde{B}_n = B_n$ and $\chi_n = -\omega_n$, for $4 \leqslant n \leqslant 3704$, $\widetilde{B}_n$ and χ_n are given in the appendix of [17].

Substituting (5.23) into (5.22) gives

$$h = \sum_{n=0}^{3704}\sum_{m=0;\,\boldsymbol{k}_n=\boldsymbol{k}_m}^{3704}(\omega_m - \chi_n)\,\mathrm{Re}\,[\widetilde{B}_m\widetilde{B}_n^*\exp[i(\chi_m - \chi_n)\,t]] + \tag{5.24a}$$
$$+ \sum_{n=0}^{3704}\sum_{m=0;\,\boldsymbol{k}_n=-\boldsymbol{k}_m}^{3704}(\omega_m + \chi_n)\,\mathrm{Re}\,[\widetilde{B}_m\widetilde{B}_n\exp[i(\chi_m + \chi_n)\,t]] + \Delta_4 + \Delta_5\,,$$

where Δ_4 and Δ_5 are terms of order $(a_0 k)^4$ and $(a_0 k)^5$, respectively, given by

$$\Delta_4 = \sum_{n=0}^{4}\sum_{m=0;\,m\neq n}^{4} T^{(2)}_{m,n,m,n}|B_m|^2|B_n|^2 + \sum_{n=0}^{4} T^{(2)}_{n,n,n,n}|B_n|^4 + \tag{5.24b}$$
$$+ (2T^{(2)}_{0,0,1,2} + T^{(2)}_{1,2,0,0} + T^{(2)}_{2,1,0,0})\,\mathrm{Re}\,[(B_0^*)^2 B_1 B_2 \exp[i\Omega_{\mathrm{I}}\,t]]\,,$$

$$\Delta_5 = (2U^{(3)}_{0,0,3,4} + U^{(2)}_{3,4,0,0,0} + U^{(2)}_{4,3,0,0,0})\,\mathrm{Re}\,[(B_0^*)^3 B_1 B_2 \exp[i\Omega_{\mathrm{II}}\,t]] + \tag{5.24c}$$
$$+ 2(U^{(3)}_{0,1,2,3,4} + U^{(3)}_{0,2,1,3,4} + U^{(3)}_{1,2,0,3,4} + U^{(3)}_{1,0,2,3,4})\,\cdot$$
$$\cdot\,\mathrm{Re}\,[B_0^* B_1^* B_2^* B_3 B_4 \exp[i(\Omega_{\mathrm{II}} - \Omega_{\mathrm{I}})\,t]] +$$
$$+ (U^{(2)}_{3,4,0,1,2} + U^{(2)}_{3,4,0,2,1} + U^{(2)}_{3,4,1,0,2} + U^{(2)}_{3,4,1,2,0} + U^{(2)}_{3,4,2,0,1} + U^{(2)}_{3,4,2,1,0} + U^{(2)}_{4,3,0,1,2} +$$
$$+ U^{(2)}_{4,3,0,2,1} + U^{(2)}_{4,3,1,2,0} + U^{(2)}_{4,3,1,0,2} + U^{(2)}_{4,3,2,0,1} + U^{(2)}_{4,3,2,1,0})\,\cdot$$
$$\cdot\,\mathrm{Re}\,[B_3^* B_4^* B_0 B_1 B_2 \exp[i(\Omega_{\mathrm{I}} - \Omega_{\mathrm{II}})\,t]]\,.$$

Note that the conditions that $\boldsymbol{k}_n = \boldsymbol{k}_m$ or $\boldsymbol{k}_n = -\boldsymbol{k}_m$ drastically reduce the number of pairs which contribute to the average energy density. The actual number of contributing wave pairs turns out to be somewhat smaller than 1000 out of a total 3705^2 possible combinations. All other wave pairs give a contribution which cancels out when averaged over the whole (x_1, x_2)-plane.

The accuracy of (5.24a) is related to the accuracy of the «amplitudes» $\widetilde{B}_n$. To obtain h accurate to order $(a_0 k)^2$, $\widetilde{B}_n$ should be accurate to order $a_0 k$, thus all the $\widetilde{B}_n$ in (5.24a) except for the first five are set to zero. One can show that the restrictions $\boldsymbol{k}_n = \pm \boldsymbol{k}_m$ exclude the possibility of products having the order $(a_0 k)^3$. This means that the result from (5.24a) obtained by using only the five free waves is accurate to the order $(a_0 k)^3$ and has an error of order $(a_0 k)^4$. We denote this result by h_3:

$$h_3 = \frac{1}{4\pi^2} \sum_{n=0}^{4} \omega_n |B_n|^2 = \frac{1}{2} g \sum_{n=0}^{4} a_n^2 . \tag{5.25}$$

For higher-order corrections one has to include bound waves. To obtain h_4, accurate to order $(a_0 k)^4$, the first 580 $\widetilde{B}_n$ are required; these include B, B', B'' and yield products of B with B'' and B' with B'. h_5 is obtained when all 3705 $\widetilde{B}_n$ are included, thus adding products of B with B''' and B' with B''. Note that in order to obtain accuracy higher than h_5 one has to add higher-order terms on the right-hand side of (2.14).

5'4. *Results.*

5'4.1. Recurrence periods. The nondimensional recurrence period $\omega_0 T$ as a function of the initial linear carrier steepness $(a_0 k)$ is shown in fig. 5 for three cases: i) class I, $\theta_{\mathrm{I}} = 0$; ii) class I, $\theta_{\mathrm{I}} = \pi/2$; iii) class II, $\theta_{\mathrm{II}} = \pi/2$. The initial phase shift in Z is given by

$$\theta = \theta_1(0) + \theta_2(0) - (J+1)\,\theta_0(0) \tag{5.26}$$

(compare with (4.8)). For all three cases we choose the relative amplitude of the initial disturbance $\mu = 0.1$ (see (5.1)). Generally speaking, the recurrence period depends on three parameters: the initial steepness of the carrier $a_0 k$, the relative amplitude of the initial disturbances μ and the initial phase difference θ. For class-I interactions STIASSNIE and KROSZYNSKI [25] obtained from the nonlinear Schrödinger equation

$$\omega_0 T = \begin{cases} 2(a_0 k)^{-2}[0.98 - 2\ln\mu - \ln|\cos\theta|]\,, & \theta \neq \pi/2\,, \\ 2(a_0 k)^{-2}[1.67 - 4\ln\mu]\,, & \theta = \pi/2\,. \end{cases} \tag{5.27}$$

The results obtained by (5.27) are represented in fig. 5 by the two low dashed straight lines, and are in fair agreement with the present results for class-I interactions. The recurrence period for class II is given in fig. 5 by the upper solid curve. The dashed line below this curve with the slope 1:3 is given for comparison, representing a relationship of the form $\omega_0 T \propto (a_0 k)^{-3}$. The dependence of the class-II recurrence period on μ and θ was found to be qualitatively

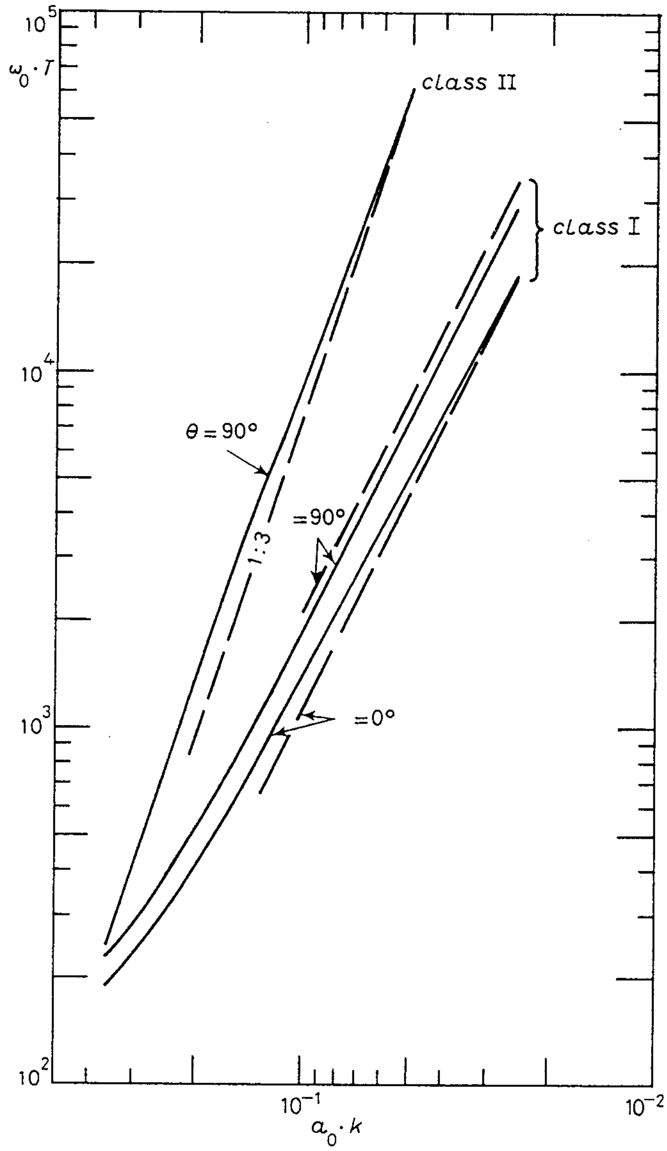


Fig. 5. – Three-wave system recurrence periods.

similar to that of class I. Namely, the periods for $\mu = 0.01$ were found to be 1.65 to 2 times greater than those for $\mu = 0.1$ (note that $\ln 0.01/\ln 0.1 = 2$, and compare with eq. (5.27)); the longest period is obtained for $\theta = \pi/2$ and the shortest for $\theta = 0$. To obtain a better physical feeling, note that the recurrence periods for very steep waves with $a_0 k = 0.36$ and $\theta = \pi/2$, $\mu = 0.1$, which are approximately equal for both classes, are 38 times the carrier period.

5'4.2. Evolution patterns. The variation in time of the amplitude of the free waves is shown in fig. 6 for class-I instability (upper row), class-II instability (middle row) and the coupled instability (lower row). The results are for three different Stokes waves having the initial steepness $a_0 k = 0.130$ (in the left column), $a_0 k = 0.227$ (middle column) and $a_0 k = 0.336$ (right column). For

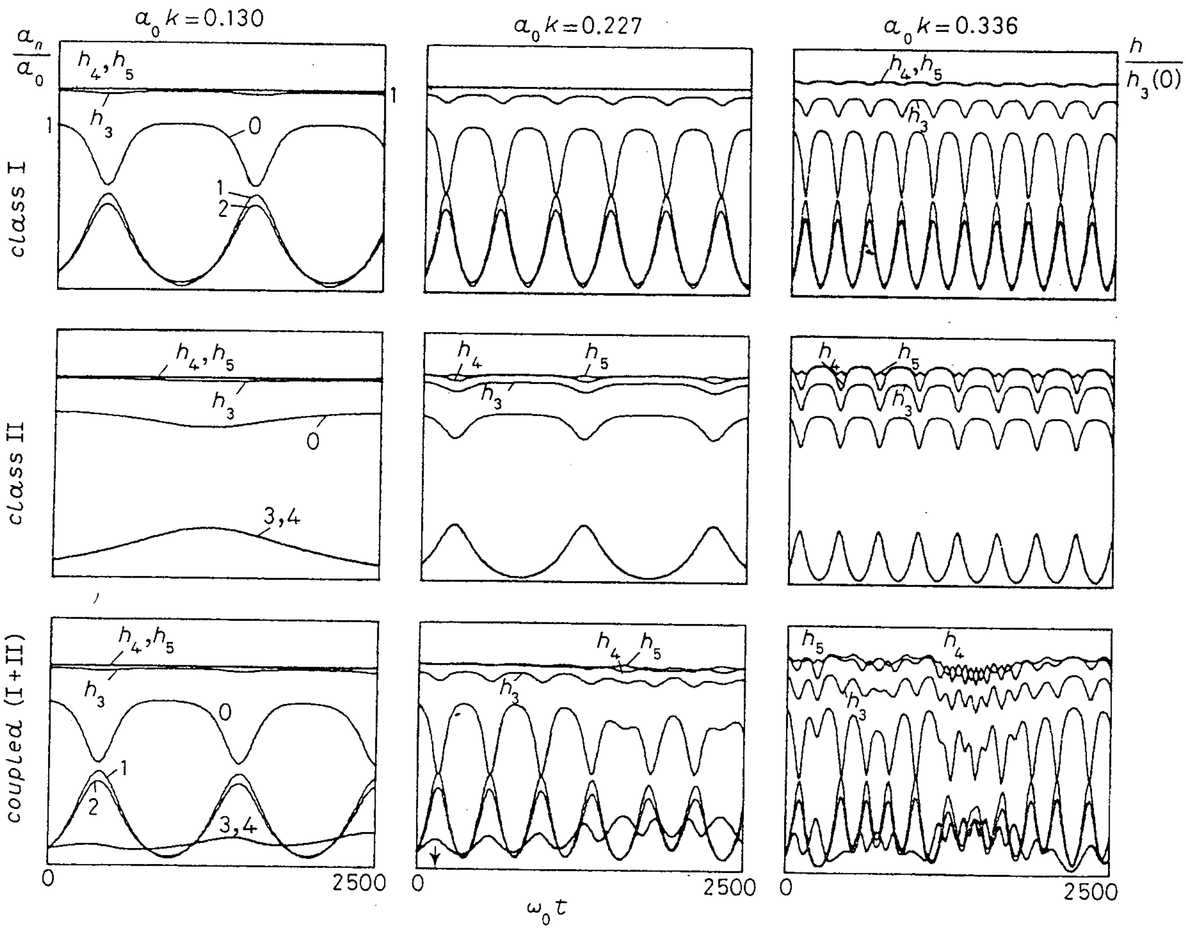


Fig. 6. – Dependence of the evolution process on the carrier steepness a_0k for $\mu_{\mathrm{I}} = \mu_{\mathrm{II}} = 0.1$ and $\theta_{\mathrm{I}} = \theta_{\mathrm{II}} = -\pi/4$.

each of these three Stokes waves most unstable class-I and class-II disturbances were introduced, see (4.5) and (4.9), defined by the following parameters:

a_0k	p_{I}	q_{II}
0.130	0.22	1.62
0.227	0.34	1.51
0.336	0.47	1.30

The curve 0 is for the carrier amplitude, the curves 1 and 2 are for the most unstable class-I disturbances, and the curves 3 and 4, which coalesce for the present problem, are for the most unstable class-II disturbances. All nine figures have a duration of about 400 carrier periods. The periodicity of both single-class evolutions and the decrease in the modulation period with the increase of the initial carrier steepness are clearly seen in this figure. DOMMERMUTH and YUE [31] used a higher-order spectral method to study the modulation of a Stokes wave train due to class-I instabilities, amongst others. They compare

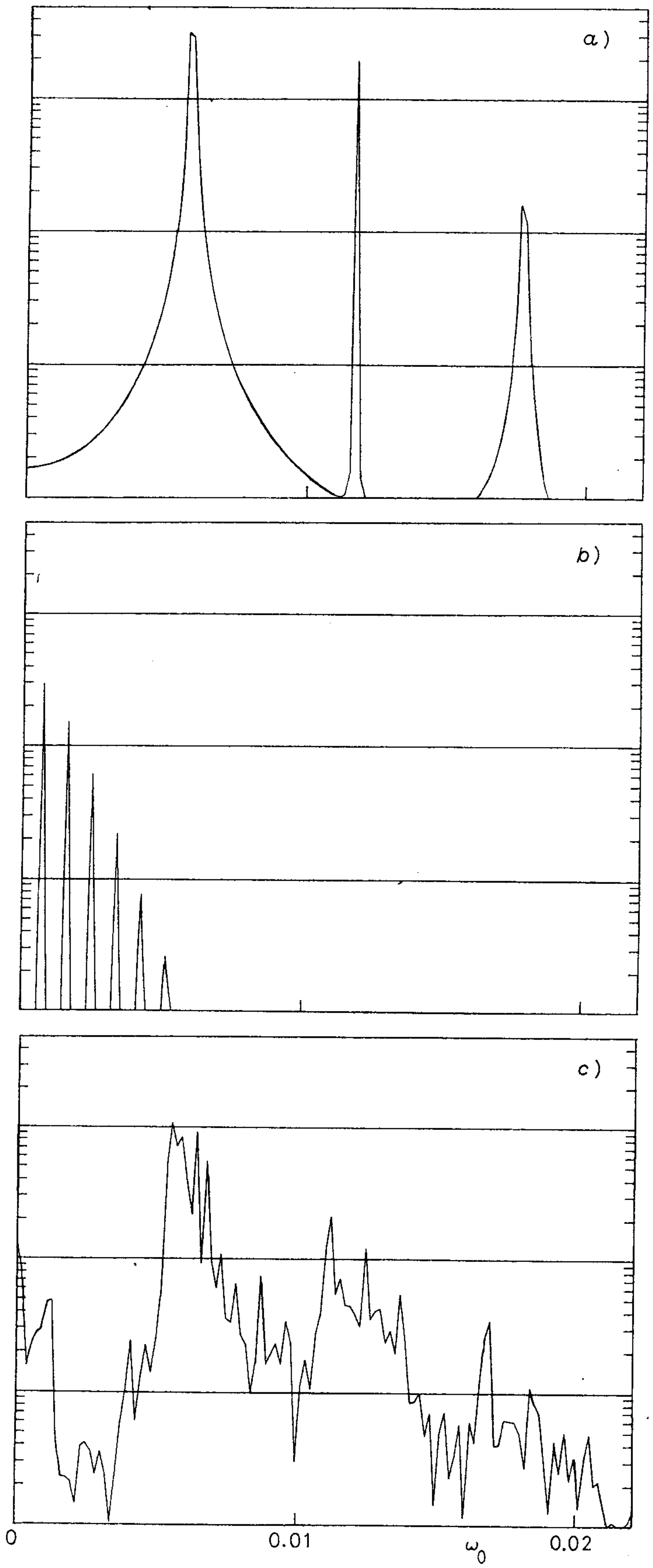


Fig. 7. – Power spectra of the amplitude of the carrier $a_0(t)$ for $a_0 k = 0.130$: *a)* class I, $\mu_I = 0.1$, $\theta_I = -\pi/4$; *b)* class II, $\mu_{II} = 0.1$, $\theta_{II} = -\pi/4$; *c)* coupled, $\mu_I = \mu_{II} = 0.1$, $\theta_I = \theta_{II} = -\pi/4$.

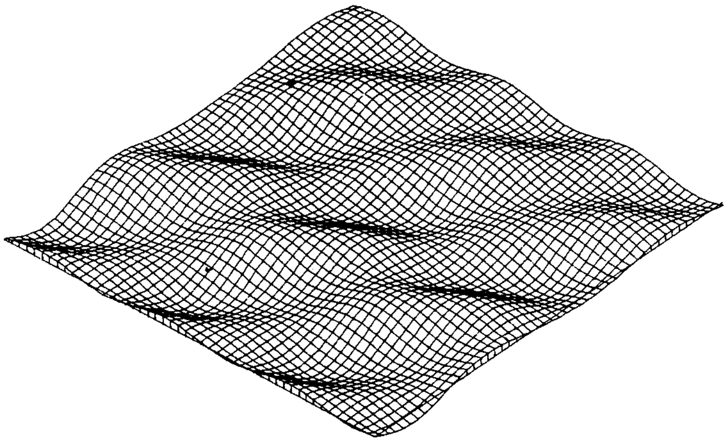


Fig. 8. – Free-surface elevation for $a_0 k = 0.227$, $\mu_{\rm I} = \mu_{\rm II} = 0.1$, $\theta_{\rm I} = \theta_{\rm II} = \pi/4$ at $\omega_0 t = 150$.

their results with ours and find that the overall qualitative behaviour is preserved.

The results for the coupled evolution are nonperiodic; this can also be seen from fig. 7, which gives the power spectra of the carrier amplitude for $a_0 k = 0.130$. We have chosen to demonstrate the results for this relatively low initial steepness, since this enables us to avoid complications involved in the

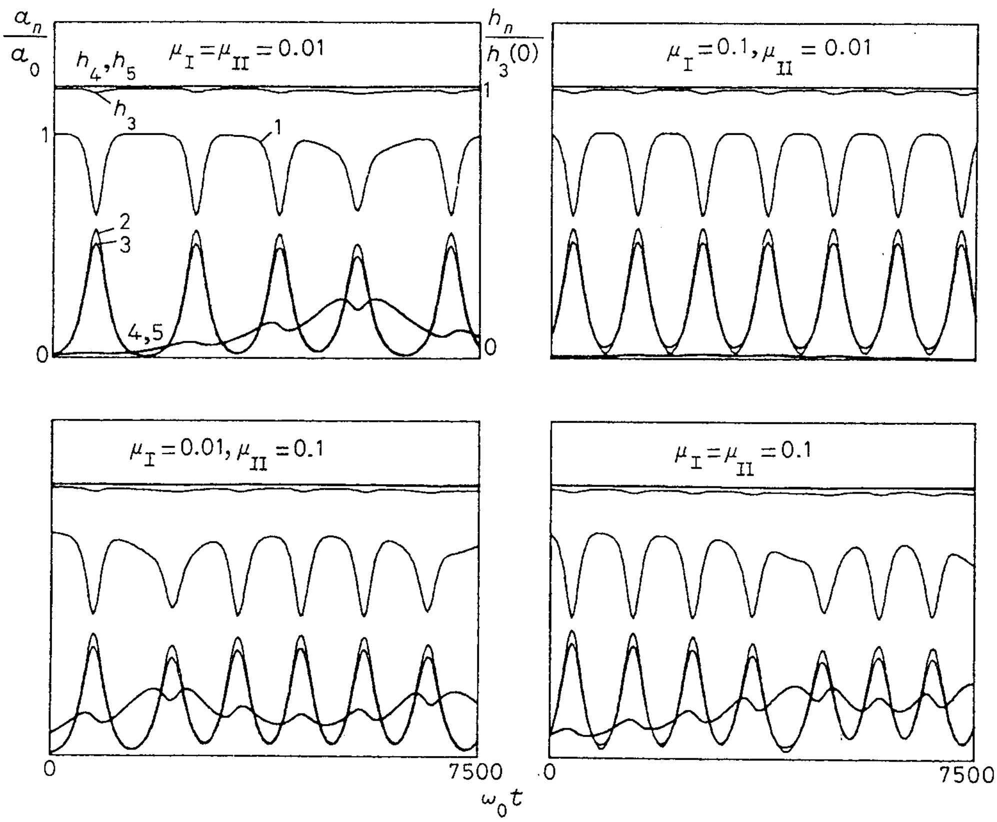


Fig. 9. – Dependence of the evolution process on the amplitudes of the initial disturbances $\mu_{\rm I}$ and $\mu_{\rm II}$ for $a_0 k = 0.130$, $\theta_{\rm I} = \theta_{\rm II} = -\pi/4$.

question of wave breaking. According to the experimental results [26], no breaking is expected as long as $a_0 k < 0.15$.

The spectra in fig. 7 were calculated from records having a duration sixty times longer than that in fig. 6. These records were divided into four equal parts, each having 1024 data points. The curves in fig. 7 are the average of four power spectra each calculated from one of these parts. The periodicity of the single-class evolutions manifests itself in the distinct equally spaced peaks in fig. 7*a*) and *b*). In fig. 7*c*) only the trace of the first class-I peak can be identified, there is no identifiable trace of the class-II peaks. The spectra for steeper waves are qualitatively similar. The above observation, as well as the fact that the amplitudes of the class-I disturbances are in general larger than those of class II, can lead to the conclusion that class-I instabilities dominate the coupled process. From the point of view of the observer of the water surface this conclusion may be somewhat misleading, since the three-dimensional class-II modes seem to catch the observer's eye more than the two-dimensional class-I modulations. This is demonstrated in fig. 8, which is a picture of the free surface at the instant marked with an arrow in the middle of the lower row of fig. 6.

In fig. 9 we present the coupled evolution for $a_0 k = 0.130$, $\theta_{\mathrm{I}} = \theta_{\mathrm{II}} = -\pi/4$, and for four different couples of initial relative amplitudes of class-I and class-II

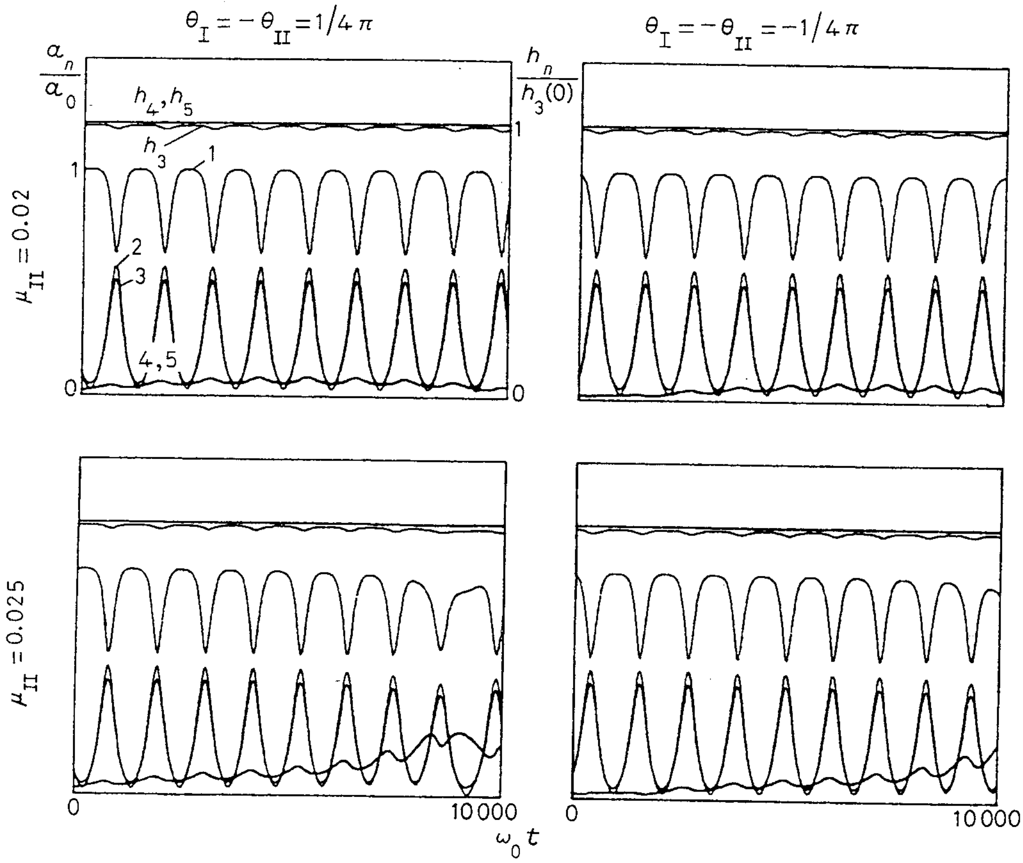


Fig. 10. – Dependence of the evolution process on the amplitude of the initial disturbance μ_{II} and the phase angles θ_{I} and θ_{II} for $a_0 k = 0.130$, $\mu_{\mathrm{I}} = 0.1$.

disturbance modes μ_{I} and μ_{II}. A general dominance of class I over class II is observed. In one case ($\mu_{\mathrm{I}} = 0.1$, $\mu_{\mathrm{II}} = 0.01$) class II is suppressed by class I throughout the evolution, which covers about 1200 carrier wave periods. A similar phenomenon appears for cases with higher initial carrier steepness. Whenever class-II disturbances take an active part, their maximum amplitude attained in the course of evolution is essentially independent of the size of the initial disturbance. On the other hand, the time required to attain this maximum depends significantly on μ_{I} and μ_{II}. The class-I dominance also manifests itself by the fact that the amplitudes of class-II modes (3, 4) oscillate with the characteristic frequency of class-I modes (1, 2).

In order to have a closer look at the parameters which influence the growth or suppression of class-II disturbances, we study the coupled evolution process for a fixed value of $\mu_{\mathrm{I}} = 0.1$ and varying values of μ_{II}, and of the initial phase shifts θ_{I} and θ_{II}. Some representative results are shown in fig. 10 (for $a_0 k = 0.130$) and in fig. 11 (for $a_0 k = 0.227$). Both figures have two columns, the left one for $\theta_{\mathrm{I}} = \pi/4$ and $\theta_{\mathrm{II}} = -\pi/4$, and the right column for $\theta_{\mathrm{I}} = -\pi/4$ and $\theta_{\mathrm{II}} = \pi/4$. These phase values are chosen since they correspond to the two possible extreme values of the initial growth rates. For $a_0 k = 0.130$ (see fig. 10), the evolution pattern does not seem to be sensitive to the initial phases, and depends primarily on μ_{II}. For $\mu_{\mathrm{II}} \leqslant 0.02$, class-II instabilities do not grow. For $\mu_{\mathrm{II}} \geqslant 0.025$, the class-II modes

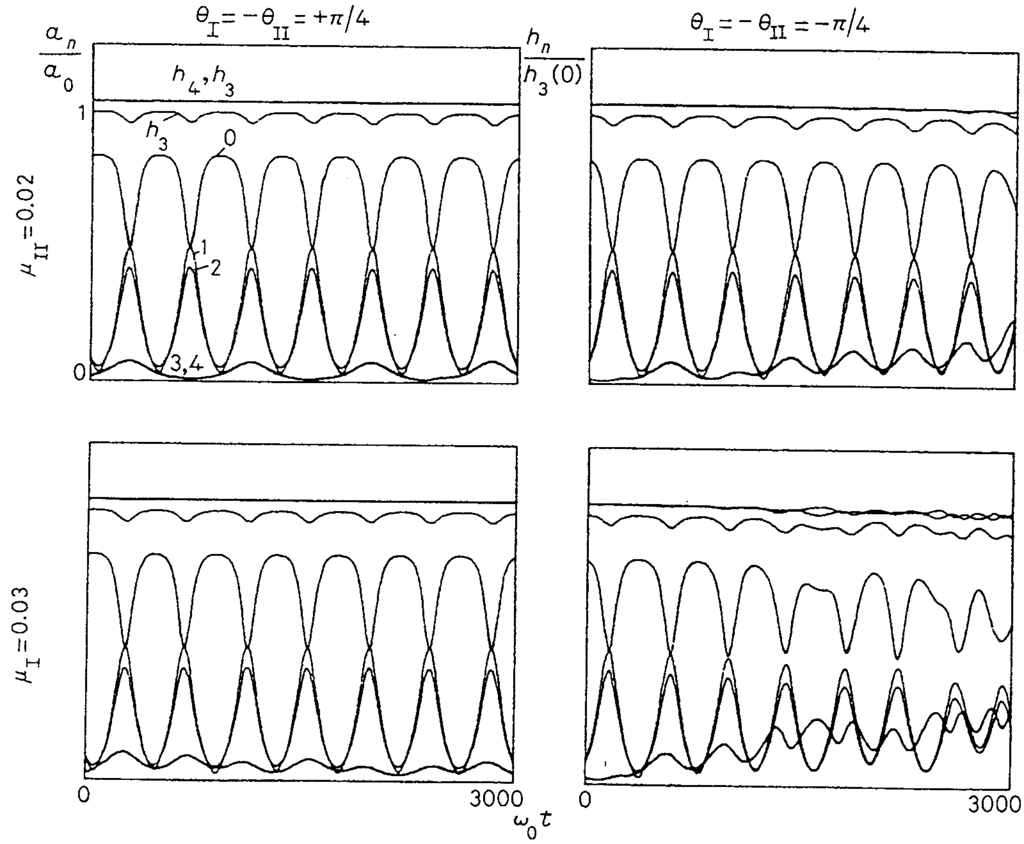


Fig. 11. – Dependence of the evolution process on the amplitude of the initial disturbance μ_{II} and the phase angles θ_{I} and θ_{II} for $a_0 k = 0.227$, $\mu_{\mathrm{I}} = 0.1$.

eventually attain their maximum value. The details for this growth depend on their initial phases; whenever class-II disturbances start growing at $t=0$ they attain their maximum faster.

For $a_0 k = 0.227$, as in the previous case, no significant class-II activity appears as long as $\mu_{\mathrm{II}} < 0.02$. For $\mu_{\mathrm{II}} > 0.02$ a profound difference between the two phases can be observed in fig. 11: for $\theta_{\mathrm{I}} = -\theta_{\mathrm{II}} = \pi/4$ class-II modes scarcely participate in the evolution process, whereas for $\theta_{\mathrm{I}} = -\theta_{\mathrm{II}} = -\pi/4$ these modes are much more active. For $\theta_{\mathrm{I}} = -\theta_{\mathrm{II}} = -\pi/4$ a small increase in μ_{II} changes the pattern significantly (see right column in fig. 11). Note that the influence of the initial phase shifts for $a_0 k = 0.227$ is opposite to that observed for $a_0 k = 0.130$; in fig. 11 class-II disturbances develop faster when they initially decrease.

5.4.3. Energy conservation. The uppermost curves in fig. 6 and 9 to 11 represent three approximations of the average energy density, *i.e.* h_3, h_4 and h_5.

Class-I interaction: one can see that the contribution of the energy terms of the order $(a_0 k)^4$ leads to a considerable improvement in the conservation of the calculated energy in the evolution process, and h_4 does not deviate practically from a horizontal line, with the exception of the highest amplitude considered. The higher-order h_5 curve does not differ from h_4.

Class-II interaction: the middle row of fig. 6 shows that the addition of the energy terms of the order $(a_0 k)^4$ changes only the «mean level» of the energy density. In order to obtain improvement in the energy conservation one has to take into account higher-order $(a_0 k)^5$ terms. Note that these terms are not necessarily positive and the h_4 and h_5 intersect.

Coupled (class I + class II) interaction: for the lowest amplitude considered $a_0 k = 0.130$, the curves h_4 and h_5 hardly differ from each other and from the horizontal straight line, giving an improvement compared to h_3. For $a_0 k = 0.227$, h_4 and h_5 give considerably better results than h_3, but some deviations from the horizontal line are seen. The deviations in h_5 are considerably smaller than those in h_4. At even higher amplitude ($a_0 k = 0.336$), h_5 is still better than h_4 but it seems that the present order of approximation is insufficient.

The present results thus indicate that the original Zakharov equation conserves energy with the relative error of $O(\varepsilon^3)$, while the modified Zakharov equation yields a relative error of $O(\varepsilon^4)$ in the averaged energy density. These results are in agreement with the conjecture that, since the Zakharov equations are approximate models of a Hamiltonian system, they conserve energy to their respective orders. The energy consideration obtained in the numerical solutions can serve as an additional confirmation of the validity of the model equations which were based on certain assumptions and involved some rather tedious algebra in their derivation. The conclusion of Yuen and Lake [2] (p. 196) that the Zakharov approximation does not conserve energy stems from the fact that they refer to h_3 (see our (5.25)) and do not take into account the higher-order approximation which includes the bound waves.

Comparison of the present results with experimental observations can shed some light on the relevance to real water waves. The available experimental results [26, 27] do not provide all the details regarding the initial noise level necessary for quantitative comparison. However, the general pattern of the theoretical results of this section is similar to their experimental observations. In all cases considered here, class-I instabilities are dominant throughout the initial stages of evolution (note that the extent of their experimental facility corresponds to our $\omega_0 t < 1000$). SU and GREEN [26] suggest that class-I modulations, which start first, trigger the class-II instability. While our approach does not support the trigger mechanism, one can see from the numerical results given in fig. 9, 10 and 11 that significant class-II activity initially appears to accompany high levels of class-I disturbances. In contrast to their reasoning, our results indicate that, whenever the initial level of class-I disturbances is substantially higher than that of class II, class-I wave components seem to suppress the three-dimensional (class II) components. These results contradict the hypothesis of the trigger mechanism. Figures 10 and 11 show that the conditions for the above suppression also include the initial phase angles of the various disturbances. For extremely steep waves the water surface becomes three-dimensional even in the initial stage (see fig. 8); this fact is in agreement with [11,12]. These experiments show that in these cases the waves break soon afterward.

6. – Concluding remarks.

In the previous sections we have presented the derivation and some applications of the modified Zakharov equation. It was shown that this equation can serve as a powerful tool in the study of irrotational water gravity waves. For example, Stokes correction to the frequency, as well as its generalization for two wave trains, was obtained in a straightforward and easy way. The same is correct regarding the stability analysis of the Stokes wave to class-I (quartet) and class-II (quintet) interacting disturbances. One can say that, once quite considerable amount of work necessary to derive the modified Zakharov equation has been invested, it allows one to obtain numerous results with relatively simple algebra. For these results to be reliable, one has to be confident about the correctness of all the interaction coefficients. This confidence can only be attained when there is a confirmation of the outcome of the model equation by results obtained by independent technique. This comparison was performed for the phase velocity shift due to the presence of a second wave train, for the instability regions of Stokes waves in water of arbitrary depth and for the location and the growth rate of the most unstable modes, as well as for the modulation periods of class-I interactions. In all these cases good agreement between the alternative approaches was attained, thus enhancing our confidence in the validity of the coefficients involved.

Several extensions of the model are possible. The most direct approach is to generalize the equation by including higher-order terms. This modification will make it possible to study class-III sextet (six waves) interactions, as well as to evaluate the Hamiltonian of the wave system to order $(a_0 k)^6$. However, we do not feel that the enormous effort necessary to obtain the higher-order interaction coefficients can be justified by the applicability of the expected results. The confidence in these coefficients will be necessarily low. Moreover, even the lower-order class-II interactions become important only for steep waves, and one can expect that extremely steep waves will be required to get significant contribution of class-III instabilities. For such waves, however, breaking cannot be disregarded any more, so that the dissipation makes the wave field non-Hamiltonian and the model less relevant.

Less cumbersome is the approach which takes into consideration the capillary effects [5, 8]. As was mentioned above, three-wave resonant interactions are possible if short enough waves (with the wavelength of the order of 1 cm and less) are considered. Such waves seem to play an important role in the wind energy transfer to the water waves. The possibility of triad interactions makes it necessary to modify the interaction coefficients. Here again, the waves considered are quite steep and dissipation can become important. JANSSEN [8] has studied the initial evolution of wind-generated gravity-capillary waves using a Zakharov-type dynamic model equation by incorporating in the equation an heuristic linear term which accounted for both dissipation and energy supply by wind. Janssen's results indicate that an alternative technique for determination of energy transfer from the wind to the waves has to be found. It appears that the problem of energy exchange between the wave field and the surrounding remains one of the major drawbacks of Zakharov-type models.

The description in the Fourier space is one of the basic features of this equation. While Fourier analysis fits well the study of interactions between various modes, it makes it difficult to consider problems with well-defined borders or those where the locations of the energy sources or sinks are given in physical space. There are, however, some exceptions which can be relatively easily treated in the general framework of Zakharov's approach. For example, when waves in an infinite tank with a rectangular cross-section are considered, a modification of Zakharov's approach which takes into account the discrete spectrum in the direction perpendicular to the channel wall has been developed [32]. To our best knowledge, there is no satisfactory way to incorporate wavemaker in the analysis based on the Zakharov equation. The only attempt in this direction was by adding a pressure disturbance at the free surface [33]. This approach can be promising in the study of naval hydrodynamics problems. However, this technique does not appear to be appropriate in many interesting cases, *e.g.* when the direct or the parametric excitation of nonlinear standing waves in a tank is considered. The nonlinear Schrödinger equation which is formulated in the physical space is in such cases a much more convenient tool (see, *e.g.*, [34, 35]). The narrow-spectrum assumption which is intrinsic to the NLS equation does not present a shortcoming for these resonant waves.

To conclude, we must repeat that the Zakharov equation is superior to all other model equations simply because it contains many of them. It does not mean, however, that it is desirable in practice to use this approach in all cases. Under certain circumstances, the selection should be in favour of less general but more convenient model equations.

* * *

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REFERENCES

[1] H. C. YUEN and B. M. LAKE: *Annu. Rev. Fluid Mech.*, **12**, 303 (1980).
[2] H. C. YUEN and B. M. LAKE: *Adv. Appl. Mech.*, **22**, 67 (1982).
[3] V. E. ZAKHAROV: *J. Appl. Mech. Tech. Phys.* (Engl. Transl.), **9**, 190 (1968).
[4] V. E. ZAKHAROV and V. G. KHARITONOV: *J. Appl. Mech. Tech. Phys.* (Engl. Transl.), **11**, 741 (1970).
[5] I. GRUMAN: M.Sc. Thesis, Technion (1987).
[6] B. CHEN and P. G. SAFFMAN: *Stud. Appl. Math.*, **72**, 125 (1985).
[7] J. ZHANG and W. K. MELVILL: *J. Fluid Mech.*, **174**, 187 (1987).
[8] P. A. E. M. JANSSEN: *J. Fluid Mech.*, **184**, 581 (1987).
[9] J. W. MCLEAN: *J. Fluid Mech.*, **114**, 315 (1982).
[10] J. W. MCLEAN: *J. Fluid Mech.*, **114**, 331 (1982).
[11] M.-Y. SU: *J. Fluid Mech.*, **124**, 73 (1982).
[12] M.-Y. SU, M. BERGIN, P. MARLER and R. MYRICK: *J. Fluid Mech.*, **124**, 45 (1982).
[13] M. S. LONGUET-HIGGINS: *Proc. R. Soc. London, Ser. A*, **360**, 489 (1978).
[14] M. STIASSNIE and L. SHEMER: *J. Fluid Mech.*, **143**, 47 (1984).
[15] S. J. HOGAN, I. GRUMAN and M. STIASSNIE: *J. Fluid Mech.*, **192**, 97 (1988).
[16] L. SHEMER and M. STIASSNIE: in *The Ocean Surface: Wave Breaking, Turbulent Mixing and Radio Probing*, edited by Y. TOBA and H. MITSUYASY (D. Reidel, Dordrecht, 1985), p. 51.
[17] M. STIASSNIE and L. SHEMER: *J. Fluid Mech.*, **174**, 299 (1987).
[18] G. G. STOKES: *Trans. Cambridge Philos. Soc.*, **8**, 441 (1849).
[19] M. S. LONGUET-HIGGINS and O. M. PHILLIPS: *J. Fluid Mech.*, **12**, 333 (1962).
[20] E. D. COKELET: *Philos. Trans. R. Soc. London, Ser. A*, **286**, 183 (1977).
[21] D. R. CRAWFORD, B. M. LAKE, P. G. SAFFMAN and H. C. YUEN: *J. Fluid Mech.*, **105**, 177 (1981).
[22] B. M. LAKE, H. C. YUEN, H. RUNGALDIER and W. E. FERGUSON: *J. Fluid Mech.*, **83**, 49 (1977).
[23] H. C. YUEN and W. E. FERGUSON: *Phys. Fluids*, **21**, 1275 (1978).

[24] H. C. YUEN and W. E. FERGUSON: *Phys. Fluids*, **21**, 2116 (1978).
[25] M. STIASSNIE and U. I. KROSZYNSKI: *J. Fluid Mech.*, **143**, 207 (1982).
[26] M.-Y. SU and A. W. GREEN: *Phys. Fluids*, **27**, 2595 (1984).
[27] M.-Y. SU and A. W. GREEN: in *The Ocean Surface: Wave Breaking, Turbulent Mixing and Radio Probing*, edited by Y. TOBA and H. MITSUYASY (D. Reidel, Dordrecht, 1985), p. 31.
[28] M. L. JAMES, G. M. SMITH and J. C. WOLFORD: *Applied Numerical Methods for Digital Computation with FORTRAN and CSMP* (Harper and Row, New York, N.Y., 1978).
[29] J. W. MILES: *J. Fluid Mech.*, **83**, 153 (1977).
[30] D. M. MILDER: *J. Fluid Mech.*, **83**, 159 (1977).
[31] D. G. DOMMERMUTH and D. K. P. YUE: *J. Fluid Mech.*, **184**, 267 (1987).
[32] T. MILOH: *Wave Motion*, **9**, 1 (1987).
[33] T. MILOH and G. DAGAN: *J. Fluid Mech.*, **159**, 433 (1985).
[34] A. F. JONES: *J. Fluid Mech.*, **138**, 53 (1984).
[35] E. KIT, L. SHEMER and T. MILOH: *J. Fluid Mech.*, **181**, 265 (1987).