The Fractal Dimension of the Ocean Surface.

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1. – Introduction.

The purpose of the present lecture is to show that the classic gravity wave problem admits stationary solutions with fractal free surfaces. The fractal dimension of these surfaces is shown to be about 2.3. In the course of making the above point one has to introduce the following assumptions:

Assume irrotational flow of an incompressible inviscid fluid.

Neglect surface tension effects and wave breaking.

Assume homogeneous wave fields.

Assume densely distributed (continuous) wave action spectrum; this leads to the requirement of uncorrelated initial random phase shifts.

Neglect the contribution of the bound components of the spectrum compared to that of the free waves.

Assume that all realizations of the free-surface elevation pass through the origin as time equals zero.

Assume that the wave field is isotropic.

Discretize the wave action spectrum using a geometric progression.

In our presentation we reverse historical order and start with Zakharov's [1] model in sect. 2, then we derive from it Hasselmann's [2] stochastic model in sect. 3, and finally relate both of them to Pierson's [3] linear model in sect. 4. In sect. 5 we present isotropic stationary solutions of Hasselmann's equation. The fractal Weierstrass-Mandelbrot function is discussed in sect. 6, and shown to fit the free surface of a homogeneous isotropic random wave field in sect. 7. Some details about the definition of dimension and the dimension of the Weierstrass function are given in the appendix.

2. – Zakharov's model.

The equations governing the irrotational flow of an incompressible inviscid fluid with a free surface and infinitely deep bottom are

$$\nabla^2\phi = 0 \qquad (z \leqslant \eta(\boldsymbol{x}, t)), \tag{2.1a}$$

$$\eta_t + (\nabla\phi \cdot \nabla\eta) - \phi_z = 0, \tag{2.1b}$$

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz = 0 \qquad (z = \eta(\boldsymbol{x}, t)), \tag{2.1c}$$

$$|\nabla\phi| \to 0 \qquad (z \to -\infty), \tag{2.1d}$$

where $\phi(\boldsymbol{x}, z, t)$ is the velocity potential, $\eta(\boldsymbol{x}, t)$ is the free-surface elevation and g the gravitational acceleration. The horizontal coordinates are $(x_1, x_2) = \boldsymbol{x}$, the vertical coordinate z is pointing upwards, and t is the time.

Given an initial condition in terms of $\eta(\boldsymbol{x}, 0)$, $\phi(\boldsymbol{x}, \eta(\boldsymbol{x}, 0), 0)$, one can transform the problem into an evolution equation in the Fourier plane

$$i\frac{\partial B}{\partial t} = I_3(\boldsymbol{k}, t) + I_4(\boldsymbol{k}, t) + \ldots. \tag{2.2}$$

The new dependent variable $B(\boldsymbol{k}, t)$ represents the free components of the wave field. $I_3, I_4, \ldots$ are integral operators representing quartet, quintet, ... nonlinear interaction, respectively.

The leading term on the r.h.s. of (2.2) was first derived by ZAKHAROV [1] and the higher-order term I_4 was obtained by STIASSNIE and SHEMER [4]:

$$I_3 = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3} B_1^* B_2 B_3 \delta_{0+1-2-3} \exp[i(\omega + \omega_1 - \omega_2 - \omega_3)t]\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3, \tag{2.3a}$$

$$I_4 = \iiiint_{-\infty}^{\infty} \{U^{(2)}_{0,1,2,3,4} B_1^* B_2 B_3 B_4 \delta_{0+1-2-3-4} \exp[i(\omega + \omega_1 - \omega_2 - \omega_3 - \omega_4)t] + \tag{2.3b}$$
$$+ U^{(3)}_{0,1,2,3,4} B_1^* B_2^* B_3 B_4 \delta_{0+1-2-3-4} \exp[i(\omega + \omega_1 + \omega_2 - \omega_3 - \omega_4)t]\}\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3\, \mathrm{d}\boldsymbol{k}_4,$$

where we use a compact notation in which the arguments $\boldsymbol{k}_i$ are replaced by the subscript i, with the subscript zero assigned to $\boldsymbol{k}$. The frequency ω is related to the wave number $\boldsymbol{k}$ through the linear dispersion relation $\omega(\boldsymbol{k}) = (g|\boldsymbol{k}|)^{1/2}$. The kernels $T^{(2)}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, $U^{(2)}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$, ..., as well as other kernels to appear in the sequel, are given in [4]. The asterisk denotes the complex conjugate.

$B(\boldsymbol{k}, t)$ is related to the Fourier transform (denoted by a hat) of $\eta(\boldsymbol{x}, t)$ and $\phi^s(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, \eta(\boldsymbol{x}, t), t)$—the velocity potential at the free surface, through

$b(\boldsymbol{k}, t)$—which is a kind of generalized «amplitude» spectrum:

$$\hat{\eta}(\boldsymbol{k}, t) = \left(\frac{k}{2\omega}\right)^{1/2} [b(\boldsymbol{k}, t) + b^*(-\boldsymbol{k}, t)], \tag{2.4a}$$

$$\hat{\phi}^s(\boldsymbol{k}, t) = -i\left(\frac{\omega}{2k}\right)^{1/2} [b(\boldsymbol{k}, t) - b^*(-\boldsymbol{k}, t)], \tag{2.4b}$$

$$b(\boldsymbol{k}, t) = [B + B' + B'' + B''' + \ldots] \exp[-i\omega(\boldsymbol{k})t]. \tag{2.4c}$$

The quantities $B', B'', \ldots$ represent the bound components of the wave field.

As an example, B' is given by

$$B' = -\iint_{-\infty}^{\infty} \left\{ V^{(1)}_{0,1,2} B_1 B_2 \delta_{0-1-2} \frac{\exp[i(\omega - \omega_1 - \omega_2)t]}{\omega - \omega_1 - \omega_2} + V^{(2)}_{0,1,2} B_1^* B_2 \delta_{0+1-2} \frac{\exp[i(\omega + \omega_1 - \omega_2)t]}{\omega + \omega_1 - \omega_2} + V^{(3)}_{0,1,2} B_1^* B_2^* \delta_{0+1+2} \frac{\exp[i(\omega + \omega_1 + \omega_2)t]}{\omega + \omega_1 + \omega_2} \right\} \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2. \tag{2.5}$$

To leading order the free-surface elevation is given by

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{k}{2\omega}\right)^{1/2} \{B(\boldsymbol{k}, t) \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)] + *\}\, \mathrm{d}\boldsymbol{k} + \text{const}. \tag{2.6}$$

The physical dimensions of the variables to be used in the sequel are

$$[B] = \mathrm{l}^{7/2}\mathrm{t}^{-1/2}, \qquad [T] = \mathrm{l}^{-3}, \qquad [\delta(\boldsymbol{k}\ldots)] = \mathrm{l}^2.$$

Note that the superscript (2) in the kernel $T^{(2)}$ has been deleted.

In the sequel we consider homogeneous wave fields which require discretized spectra for their representation:

$$B(\boldsymbol{k}, t) = \sum_n B_n(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}_n), \qquad \boldsymbol{k}_n \neq 0, \tag{2.7}$$

so that (2.6) and Zakharov's equations are, respectively, replaced by

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi} \sum_n \left(\frac{k_n}{2\omega_n}\right)^{1/2} \{B_n(t) \exp[i(\boldsymbol{k}_n\cdot\boldsymbol{x} - \omega_n t)] + *\} + \text{const}, \tag{2.8}$$

$$i\frac{\mathrm{d}B_n}{\mathrm{d}t} = \sum_{p,q,r} T_{n,p,q,r}\, \delta_{npqr} \exp[i(\omega_n + \omega_p - \omega_q - \omega_r)t]\, B_p^* B_q B_r. \tag{2.9}$$

Note that $[B_n] = l^{3/2} t^{-1/2}$; and δ_{npqr} an abbreviated form of Kronecker's delta, $\delta_{n+p,q+r}$, is dimensionless.

Multiplying (2.9) by $-iB_n^*$ and adding to the result its complex conjugate gives

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_n|^2 = 2\,\mathrm{Re}\left\{\frac{1}{i}\sum_{p,q,r} T_{npqr}\delta_{npqr}\exp[i\Delta_{npqr}t]\,B_n^* B_p^* B_q B_r\right\}, \tag{2.10}$$

where $\Delta_{npqr} = \omega_n + \omega_p - \omega_q - \omega_r$, $\omega_n^2 = g|\boldsymbol{k}_n|$.

For the derivation of Zakharov's equation we recall that only terms for which $\Delta_{npqr}/\omega_n = o(1)$, namely, nearly resonating quartets, contribute significantly in eqs. (2.9), (2.10). For near-resonance conditions one can show that $T_{npqr} = T_{nprq} \simeq T_{pnqr} \simeq T_{qrnp}$. All the $\simeq$ signs become $=$ for exact resonance conditions.

3. – Hasselmann's stochastic model.

Here it is assumed that the number of components tends to infinity, so that in the limit they become densely distributed over the relevant domain in the wave number plane.

It seems that, if one wants to stick to the above assumption and still remain in a reasonable physical framework, he finds it necessary to take the phases of the components however close to each other, uncorrelated to lowest order.

Now, since the phases of the B_n's are assumed to be nearly uncorrelated, the product $B_n^* B_p^* B_q B_r$ will, on the average, be negligible except when either $p = q$, $n = r$ or else $n = q$, $p = r$. Hence the term in the curly brackets in (2.10) reduces to $-2i|B_n|^2 \sum_r T_{nrnr}|B_r|^2$, which is imaginary. Thus the r.h.s. of (2.10) vanishes to lowest order. To calculate higher-order terms, we first differentiate the product $B_n^* B_p^* B_q B_r$ with respect to t and substitute from (2.9)

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}(B_n^* B_p^* B_q B_r) = &-B_p^* B_q B_r \sum_{u,v,w} T_{nuvw}\delta_{nuvw}\exp[-i\Delta_{nuvw}t]\,B_u B_v^* B_w^* - \\ &- B_n^* B_q B_r \sum_{u,v,w} T_{puvw}\delta_{puvw}\exp[-i\Delta_{puvw}t]\,B_u^* B_v^* B_w^* + \\ &+ B_n^* B_p^* B_r \sum_{u,v,w} T_{quvw}\delta_{quvw}\exp[i\Delta_{quvw}t]\,B_u^* B_v B_w + \\ &+ B_n^* B_p^* B_q \sum_{u,v,w} T_{ruvw}\delta_{ruvw}\exp[i\Delta_{ruvw}t]\,B_u^* B_v B_w . \end{aligned} \tag{3.1}$$

The contributions of most of the terms in the above equation cancel out on the average, thus we obtain

$$\begin{aligned} &i\frac{\mathrm{d}}{\mathrm{d}t}(B_n^* B_p^* B_q B_r) = \\ &= -2T_{npqr}\delta_{npqr}\exp[-i\Delta_{npqr}t]\,[C_q C_r(C_p + C_n) - C_n C_p(C_q + C_r)], \end{aligned} \tag{3.2}$$

where

(3.3) $$C_n = |B_n|^2$$

is the wave action spectrum.

Integrating eq. (3.2) with respect to t from $-\infty$ (where the correlations are assumed negligible) up to t yields

(3.4) $$B_n^* B_p^* B_q B_r = \\ = 2iT_{npqr}\delta_{npqr}[C_q C_r(C_p + C_n) - C_n C_p(C_q + C_r)] \int_{-\infty}^{t} \exp[-i\Delta_{npqr}\tau]\,d\tau;$$

the factor in square brackets has been assumed to vary slowly compared to the exponent, so that it was taken outside the integral.

Substituting into (2.10) yields

(3.5) $$\frac{dC_n}{dt} = \\ = 4\sum_{p,q,r} T^2_{npqr}\delta_{npqr}[C_q C_r(C_p + C_n) - C_n C_p(C_q + C_r)]\,\mathrm{Re}\int_{-\infty}^{t} \exp[i\Delta_{npqr}(t-\tau)]\,d\tau.$$

Since $\mathrm{Re}\int_{-\infty}^{0} \exp[-i\Delta s]\,ds = \pi\delta(\Delta)$, we finally obtain

(3.6) $$\frac{dC_n}{dt} = 4\pi\sum_{p,q,r} T^2_{npqr}\delta_{npqr}\delta(\omega_n + \omega_p - \omega_q - \omega_r)[C_q C_r(C_p + C_n) - C_n C_p(C_q + C_r)].$$

This discretized version of Hasselmann's model is rewritten in integral notation as follows:

(3.7) $$\frac{\partial C(k,t)}{\partial t} = 4\pi \iiint_{-\infty}^{\infty} T^2(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\delta(\omega+\omega_1-\omega_2-\omega_3)\,\cdot \\ \cdot [C(\boldsymbol{k}_2)\,C(\boldsymbol{k}_3)\cdot(C(\boldsymbol{k}_1)+C(\boldsymbol{k})) - C(\boldsymbol{k})\,C(\boldsymbol{k}_1)\cdot(C(\boldsymbol{k}_2)+C(\boldsymbol{k}_3))]\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\,d\boldsymbol{k}_3,$$

so that (3.6) is obtained from (3.7) when

(3.8) $$C(\boldsymbol{k},t) = \sum_n C_n(t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_n).$$

Note that $[C_n] = \mathrm{l}^3\mathrm{t}^{-1}$ and that $[C] = \mathrm{l}^5\mathrm{t}^{-1}$.

The arguments of B_n vary on a faster time scale than $|B_n|$. Indeed, from (2.9) one can show that on the average

(3.9) $$\frac{d}{dt}(\arg B_n) = -\sum_p e_{np} T_{n,p,n,p}|B_p|^2 = -\Omega_n,$$

where $e_{np} = 1$ for $n = p$ and $e_{np} = 2$ for $n \neq p$.

Thus

$$\arg B_n = \varepsilon_n - \Omega_n t\,, \tag{3.10}$$

where ε_n is an initial random phase, assumed to have a rectangular distribution in the range $(-\pi, \pi)$. Ω_n is the Stokes correction of the frequency. From (2.5) and (2.7) it follows that the wave action spectrum of the bound components $C'_m = |B'_m|^2$ is given by

$$C'_m = \sum_{p,q} C_p C_q \left\{ e_{pq} (V^{(1)}_{m,p,q})^2 \frac{\delta_{m,p+q}}{(\omega_m - \omega_p - \omega_q)^2} + (V^{(2)}_{m,p,q})^2 \frac{\delta_{m,q-p}}{(\omega_m + \omega_p - \omega_q)^2} + \right.$$
$$\left. + e_{pq} (V^{(3)}_{m,p,q})^2 \frac{\delta_{m,-p-q}}{(\omega_m + \omega_p + \omega_q)^2} \right\}. \tag{3.11}$$

4. – Pierson's linear model.

From (2.4*a*), (3.3) and (3.10) the free-surface elevation is given by

$$\eta = \frac{1}{\pi} \sum_n \left(\frac{k_n}{2\omega_n} \right)^{1/2} \sqrt{C_n} \cos(\boldsymbol{k}_n \cdot \boldsymbol{x} - \tilde{\omega}_n t + \varepsilon_n) + \text{const}\,, \tag{4.1}$$

where

$$\tilde{\omega}_n = \omega_n + \Omega_n\,, \tag{4.2}$$

and Ω_n is given by (3.9). The relatively small contributions of the bound components given by (3.11) to the free-surface elevation have been neglected.

In order to facilitate the comparison (in sect. 7) with the Weierstrass-Mandelbrot function, the constant in (4.1) is chosen so that all realizations of $\eta(t=0)$ pass through the origin, giving

$$\eta = \frac{1}{\pi} \sum_n \left(\frac{k_n}{2\omega_n} \right)^{1/2} \sqrt{C_n} \{\cos(\boldsymbol{k}_n \cdot \boldsymbol{x} - \tilde{\omega}_n t + \varepsilon_n) - \cos\varepsilon_n\}\,. \tag{4.3}$$

Switching back to the continuous notation, (4.3) becomes

$$\eta = \int_{-\infty}^{\infty} \{\cos(\boldsymbol{k} \cdot \boldsymbol{x} - \tilde{\omega} t + \varepsilon) - \cos\varepsilon\} \left(\frac{|\boldsymbol{k}|}{2\pi^2 \omega} C(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k} \right)^{1/2}, \tag{4.4}$$

where from (3.9)

$$\tilde{\omega} = \sqrt{g|\boldsymbol{k}_0|} + \int_{-\infty}^{\infty} e_{01} T_{0101} C_1\, \mathrm{d}\boldsymbol{k}_1\,. \tag{4.5}$$

The quantity $(2\pi)^{-2}|\boldsymbol{k}|\,C(k)/\omega = \psi(\boldsymbol{k})$, namely the two-dimensional wave number energy spectrum, is usually specified in terms of the covariance of the surface displacement at points separated by a distance $\boldsymbol{r}$

$$\psi(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \overline{\eta(\boldsymbol{x})\,\eta(\boldsymbol{x}+\boldsymbol{r})}\,\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]\,\mathrm{d}\boldsymbol{r}\,. \tag{4.6}$$

PIERSON has shown that (4.4), or actually its discretized realization (4.3), represents a multivariate Gaussian process, stationary in the variables $\boldsymbol{x}$, t. Note that Pierson's original linear model did not take into account the nonlinear Stokes correction of the frequency.

5. – Stationary solutions of Hasselmann's equation.

ZAKHAROV and ZASLAVSKIY [5] found that (3.7) has two isotropic stationary solutions

$$C \propto |\boldsymbol{k}|^{-\beta}\,, \qquad \beta = 4\,,\ 3\frac{5}{6}\,. \tag{5.1}$$

Here we use a somewhat different method and prove that for the one-dimensional stationary case only one solution of (3.7) exists and that it is given by

$$C \propto |k|^{-\beta}\,, \qquad \beta = 2\frac{5}{6}\,. \tag{5.2}$$

Starting from Hasselmann's model,

$$\frac{\partial C}{\partial t} = 4\pi \iiint_{-\infty}^{\infty} T^2_{0,1,2,3}\,[C_2C_3(C_1+C) - CC_1(C_2+C_3)]\,\delta_{0+1-2-3}\,\delta^{\omega}_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,, \tag{3.7}$$

where C is the wave action spectrum and δ is Dirac's delta-function. The kernel T^2 being the square of T for strict resonance conditions has the following symmetries

$$T_{0,1,2,3} = T_{0,1,3,2} = T_{1,0,2,3} = T_{2,3,0,1} \tag{5.3}$$

and is a homogeneous function of order 6, since

$$T_{\alpha k,\alpha k_1,\alpha k_2,\alpha k_3} = |\alpha|^3\, T_{k,k_1,k_2,k_3}\,. \tag{5.4}$$

For the degenerate one-dimensional case the vector wave numbers in (3.7) are replaced by scalar wave numbers. Integrating (3.7) with respect to k_1 and then changing variables $k_2 \to k_1$, $k_3 \to k_2$ yields

$$(5.5)\quad \frac{\partial C}{\partial t} = \\ = 4\pi \iint_{-\infty}^{\infty} T^2_{0,1+2-0,1,2}[C_1 C_2(C_{1+2-0}+C) - CC_{1+2-0}(C_1+C_2)]\,\delta^{\omega}_{0,1+2-0,1,2}\,\mathrm{d}k_1\,\mathrm{d}k_2 .$$

Since the next step is the integration of (5.5) with respect to k_2, we write

$$(5.6)\qquad \delta^{\omega}_{0,1+2-0,1,2} = \delta\{\sqrt{|k|} + \sqrt{|k_1+k_2-k} - \sqrt{|k_1|} - \sqrt{|k_2|}\} = \delta\{f(k_2)\},$$

which is a function of k_2 with k, k_1 as parameters.

According to exercise 32, p. 285 in [6],

$$(5.7)\qquad \delta\{f(k_2)\} = \sum_m \frac{\delta(k_2 - K_m)}{|f'(K_m)|},$$

where K_m are the roots of $f(k_2)=0$.

Without loss of generality I am considering the case $k>0$. The equation $f(k_2)=0$ has four roots, which are designated by a, b, c and d:

$$(5.8a)\quad k>k_1>0, \quad k_a = \left[\sqrt{k} - \sqrt{k_1} + \sqrt{(\sqrt{k}-\sqrt{k_1})^2 - 4(k_1 - \sqrt{kk_1})}\right]^2/4 > 0, \quad k_1 + k_a - k < 0,$$

$$(5.8b)\quad k_1>k, \quad k_b = -\left[\sqrt{k} - \sqrt{k_1} + \sqrt{(\sqrt{k_1}-\sqrt{k})^2 - 4(k - \sqrt{kk_1})}\right]^2/4 < 0, \quad k_1 + k_b - k > 0,$$

$$(5.8c)\quad 0>k_1>-k, \quad k_c = [(k - k_1 - \sqrt{-kk_1})/(\sqrt{k} - \sqrt{-k_1})]^2 > 0, \quad k_1 + k_c - k > 0,$$

$$(5.8d)\quad -k>k_1, \quad k_d = kk_1/(\sqrt{-k_1} - \sqrt{k})^2 < 0, \quad k_1 + k_d - k < 0.$$

The weight functions $g_m = |f'(K_m)|$ are

$$(5.9a)\qquad g_a = \sqrt{(\sqrt{k}-\sqrt{k_1})^2 + 4(\sqrt{kk_1} - k_1)}/2(\sqrt{kk_1} - k_1),$$

$$(5.9b)\qquad g_b = \sqrt{(\sqrt{k_1}-\sqrt{k})^2 + 4(\sqrt{kk_1} - k_1)}/2(\sqrt{kk_1} - k),$$

$$(5.9c)\qquad g_c = (\sqrt{k} - \sqrt{-k_1})^3/2\sqrt{-kk_1}\,(k - k_1 - \sqrt{-kk_1}),$$

$$(5.9d)\qquad g_d = (\sqrt{-k_1} - \sqrt{k})^3/2\sqrt{-kk_1}\,(k - k_1 - \sqrt{-kk_1}).$$

Thus

$$\frac{\partial C}{\partial t} = 4\pi \int_{-\infty}^{-k} \frac{\mathrm{d}k_\delta}{g_d} T^2_{0,\delta+d-0,\delta,d}[C_\delta C_d(C_{\delta+d-0}+C) - CC_{\delta+d-0}(C_\delta + C_d)] + \tag{5.10}$$

$$+ 4\pi \int_{-k}^{0} \frac{\mathrm{d}k_1}{g_c} T^2_{0,1+c-0,1,c}[C_1 C_c(C_{1+c-0}+C) - CC_{1+c-0}(C_1 + C_c)] +$$

$$+ 4\pi \int_{0}^{k} \frac{\mathrm{d}k_1}{g_a} T^2_{0,1+a-0,1,a}[C_1 C_a(C_{1+a-0}+C) - CC_{1+a-0}(C_1 + C_a)] +$$

$$+ 4\pi \int_{k}^{\infty} \frac{\mathrm{d}k_\beta}{g_b} T^2_{0,\beta+b-0,\beta,b}[C_\beta C_b(C_{\beta+b-0}+C) - CC_{\beta+b-0}(C_\beta + C_b)] .$$

The domain b is mapped into the domain a, and the domain d is mapped into c, by the following change of variables:

$$k_\beta = k^2/k_1 , \qquad k_\delta = k^2/k_1 . \tag{5.11}$$

Applying these transformations to the various terms in the integrands of (5.10) gives

$$\begin{cases} k_\beta = (k/k_1)\,k , \quad k = (k/k_1)\,k_1, \\ \qquad k_b = (k/k_1)(k_1 + k_a - k) , \quad k_\beta + k_b - k = (k/k_1)\,k_a ; \\ k_\delta = (k/k_1)\,k , \quad k = (k/k_1)\,k_1 , \\ \qquad k_d = (k/k_1)(k_1 + k_c - k) , \quad k_\delta + k_d - k = (k/k_1)\,k_c . \end{cases} \tag{5.12}$$

From (5.12), (5.3) and (5.4) one can show that

$$T^2_{0,\beta+b-0,\beta,b} = (k/k_1)^6 T^2_{0,1+a-0,1,a} , \qquad T^2_{0,\delta+d-0,\delta,d} = (k/k_1)^6 T^2_{0,1+c-0,1,c} . \tag{5.13}$$

Equations (5.12) and (5.9) give

$$g_b = (k_1/k)^{1/2} g_a , \qquad g_d = (-\,k_1/k)^{1/2} g_c . \tag{5.14}$$

Substitution of the above into (5.10) yields

$$(4\pi)^{-1}\frac{\partial \overset{(+)}{C}(k)}{\partial t} = \int_{-k}^{0} \frac{\mathrm{d}k_1}{g_c} T^2_{0,1+c-0,1,c}\Big\{[\overset{(-)}{C}(k_1)\overset{(+)}{C}(k_c)(\overset{(+)}{C}(k_1+k_c-k) + \overset{(+)}{C}(k)) - \tag{5.15}$$

$$- \overset{(+)}{C}(k)\overset{(+)}{C}(k_1+k_c-k)(\overset{(-)}{C}(k_1) - \overset{(+)}{C}(k_c))] + \left(-\frac{k}{k_1}\right)^{8.5}\left[\overset{(-)}{C}\left(\frac{k^2}{k_1}\right)\overset{(-)}{C}\left(\frac{k(k_1+k_c-k)}{k_1}\right) .\right.$$

$$\cdot\left(\overset{(-)}{C}\left(\frac{kk_c}{k_1}\right)+\overset{(+)}{C}(k)\right)-\overset{(+)}{C}(k)\overset{(-)}{C}\left(\frac{kk_c}{k_1}\right)\left(\overset{(-)}{C}\left(\frac{k^2}{k_1}\right)+\overset{(-)}{C}\left(\frac{k(k_1+k_c-k)}{k_1}\right)\right)\Bigg]\Bigg\}+$$

$$+\int_0^k \frac{dk_1}{g_a}T^2_{0,1+a-0,1,a}\Big\{[\overset{(+)}{C}(k_1)\overset{(+)}{C}(k_a)(\overset{(-)}{C}(k_1+k_a-k)+\overset{(+)}{C}(k))-$$

$$-\overset{(+)}{C}(k)\overset{(-)}{C}(k_1+k_a-k)(\overset{(+)}{C}(k_1)+\overset{(+)}{C}(k_a))]+\left(\frac{k}{k_1}\right)^{8.5}\left[\overset{(+)}{C}\left(\frac{k^2}{k_1}\right)\overset{(-)}{C}\left(\frac{k(k_1+k_a-k)}{k_1}\right)\cdot\right.$$

$$\cdot\left(\overset{(+)}{C}\left(\frac{kk_a}{k_1}\right)+\overset{(+)}{C}(k)\right)-\overset{(+)}{C}(k)\overset{(+)}{C}\left(\frac{kk_a}{k_1}\right)\left(\overset{(+)}{C}\left(\frac{k^2}{k_1}\right)+\overset{(-)}{C}\left(\frac{k(k_1+k_a-k)}{k_1}\right)\right)\Bigg]\Bigg\}.$$

The plus and minus signs above the C's indicate the direction of propagation of that component. Equation (5.15) has a stationary solution of the form

$$C(k)\propto |k|^s. \tag{5.16}$$

Substitution of (5.16) into the expressions in the curly brackets in both integrands of (5.15) gives

$$\{*\}=[|k_1|^s|k_f|^s(|k_1+k_f-k|^s+|k|^s)-|k|^s|k_1+k_f-k|^s(|k_1|^s+|k_f|^s)]+$$
$$+\left|\frac{k}{k_1}\right|^{8.5}\left[\left|\frac{k^2}{k_1}\right|^s\left|\frac{k(k_1+k_f-k)}{k_1}\right|^s\left(\left|\frac{kk_f}{k_1}\right|^s+|k|^s\right)-\right.$$
$$\left.-|k|^s\left|\frac{kk_f}{k_1}\right|^s\left(\left|\frac{k^2}{k_1}\right|^s+\left|\frac{k(k_1+k_f-k)}{k_1}\right|^s\right)\right], \tag{5.17}$$

where k_f is either k_c of k_a.

$S=-17/6$ gives $\{*\}=0$. Thus we have found a stationary solution of the one-dimensional energy transfer equation

$$C(k)\propto |k|^{-17/6}. \tag{5.2}$$

6. – The Weierstrass-Mandelbrot function.

The univariate Weierstrass-Mandelbrot function $W(x_1)$ is a superposition of sinusoids with geometrically spaced wave numbers, and amplitudes that follow a power law. It is given by

$$W(x_1)=\sum_{n=-\infty}^{\infty}\gamma^{-n(2-D)}(1-\exp[i\gamma^n x_1])\exp[i\phi_n], \tag{6.1}$$

where $\gamma>1$, $1<D<2$ and the phases ϕ_n are arbitrary. For the deterministic case the phases ϕ_n are chosen by a special rule; for the stochastic case we will choose the ϕ_n as independent random variables uniformly distributed between $-\pi$ and π. Each choice of the ϕ_n gives another member of the ensemble of stochastic functions $W(x)$. By choosing the phases to be independent, and letting $\gamma\to 1^+$, $W(x_1)$ is made to be a Gaussian random function. Note that the sum defining $W(x_1)$ extends from $-\infty$ to $+\infty$. This means that the wave numbers γ^n extend from 0 to ∞. It is in this sense that there is neither a larger scale nor a smaller scale of variation of $W(x_1)$ with x_1. The continuation of the sum to $-\infty$ and the insertion of 1 in the numerator of the summand is Mandelbrot's addition to the Weierstrass function. Extension of the sum to $-\infty$ ensures perfect scaling. The 1 in the summand as well as the condition $1<D<2$ are required for convergence of the sum. If the sum were to terminate at some smaller value of n, say n_{min}, and extended to $+\infty$ we would require only that $D<2$.

D is known as the fractal dimension of the graph $W(x_1)$; by which we mean, since W is complex, the graph of $\operatorname{Re} W$ or $\operatorname{Im} W$. With the indicated restriction on γ and D, the series for W converges but the series for dW/dx_1 does not.

A generalization of the univariate W.M. function was given by AUSLOOS and BERMAN [7]:

$$W(\boldsymbol{x}) = (\ln\gamma/M)^{1/2}\sum_{m=1}^{M} A_m \sum_{n=-\infty}^{\infty} (k_0\gamma^n)^{D-3}\cdot \\ \cdot\{1-\exp[i[k_0\gamma^n(x_1\cos\theta_m + x_2\sin\theta_m)]]\}\exp[i(\phi_{mn})]. \tag{6.2}$$

Here $\gamma>1$, the θ_m are equally spaced over $(-\pi,\pi)$, the amplitudes A_m are chosen in a deterministic way, and ϕ_{mn} are either deterministic or random. The value of M is discussed in the following section. Most important of all, D, which is in the range $2<D<3$, to ensure convergence, is believed to be the fractal dimension of $W(x_1, x_2)$.

The real part of $W(\boldsymbol{x})$, slightly modified by the addition of time-dependent deterministic phase shifts, is

$$\operatorname{Re}\{W(\boldsymbol{x})\} = (\ln\gamma/M)^{1/2}\sum_{m=1}^{M} A_m \sum_{n=-\infty}^{\infty} (k_0\gamma^n)^{D-3}\cdot \\ \cdot\{\cos\phi_{mn} - \cos[k_0\gamma^n(x_1\cos\theta_m + x_2\sin\theta_m) - \sqrt{gk_0}\,\gamma^{n/2}t + \phi_{mn}]\}. \tag{6.3}$$

For isotropic cases the A_m's are all equal. More details about the definition of dimension and about the fractal dimension of the Weierstrass function are given in the appendix.

7. – The fractal dimension of the free-surface elevation.

The random free surface of a homogeneous isotropic and stationary ocean is given from (4.3) and (5.1) by

$$(7.1)\qquad \eta = \frac{1}{\pi}\sum_{m=1}^{M}\sum_{n}\left(\frac{k_n}{2\omega_n}\right)^{1/2}\sqrt{C_{m,n}}\cdot \cdot\left\{\cos\left[k_n\left(x_1\cos\frac{2\pi m}{M}+x_2\sin\frac{2\pi m}{M}\right)-\tilde{\omega}_n t+\varepsilon_{m,n}\right]-\cos\left[\varepsilon_{m,n}\right]\right\},$$

where $C_{m,n}$ is a discretized form of the continuous wave action spectrum,

$$(7.2)\qquad C = C_0(k_0/k)^\beta, \qquad \beta=4,\quad 3\frac{5}{6}.$$

M and n tend to infinity so that $k_{m,n}=k_n(\cos(2\pi m/M), \sin(2\pi m/M))$ is densely distributed over the whole wave number domain.

The straightforward way to specify k_n is probably

$$(7.3)\qquad k_n = n\,\Delta k, \qquad n=0, 1, 2, \ldots,\ \Delta k\to 0.$$

Nevertheless it is rather rewarding, and equally legitimate, to replace the arithmetic progression (7.3) by the following geometric series:

$$(7.4)\qquad k_n = k_0\gamma^n, \qquad n=\ldots -2, -1, 0, 1, 2, \ldots,\ \gamma\to 1^+.$$

From (7.2) and (7.4) we have

$$(7.5)\qquad C_{m,n}=\frac{2\pi}{M}C(k_n)\,k_n^2\ln\gamma=\frac{2\pi\ln\gamma}{M}C_0^\beta k_0^2\gamma^{2n-\beta}.$$

Substituting (7.5) into (7.1) yields

$$(7.6)\qquad \eta=\frac{k_0^{5/4}}{g^{1/4}}\left(\frac{C_0\ln\gamma}{\pi M}\right)^{1/2}\sum_{m=1}^{M}\sum_{n=-\infty}^{\infty}\gamma^{(5-2\beta)n/4}\cdot \cdot\left\{\cos\left[k_0\gamma^n\left(x_1\cos\frac{2\pi m}{M}+x_2\sin\frac{2\pi m}{M}\right)-\tilde{\omega}_n t+\varepsilon_{m,n}\right]-\cos\left[\varepsilon_{m,n}\right]\right\}.$$

Comparing (7.6) with the W.M. function (6.3), one can see that $D-3=(5-2\beta)/4$, so that

$$(7.7)\qquad D=4.25-0.5\beta=\begin{cases}2.25 & \text{for } \beta=4,\\[4pt] 2\frac{1}{3} & \text{for } \beta=3\frac{5}{6}.\end{cases}$$

The above result is an indication of the possibility that the free surface of the ocean in appropriate circumstances can become a fractal with dimension of about 2.3.

For the one-dimensional case, where waves moving to the left have the same amplitudes as those moving to the right, (5.2) and (6.1) yield $D = 4/3$.

* * *

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APPENDIX

Definition of dimension.

In this appendix we define and discuss two somewhat different concepts of dimension, the capacity and the Hausdorff dimension; both require only a metric (*i.e.* a concept of distance) for their definition. The term «fractal dimension» used in the heading of the present lecture was originally coined by MANDELBROT who used it as a synonym for Hausdorff dimension. Other authors use the term «fractal dimension» as a synonym for capacity. Nevertheless, for many examples the capacity and Hausdorff dimension take on a common value.

Capacity. – The capacity of a set was originally defined by KOLMOGOROV [8]. It is given by

$$d_c = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log (1/\varepsilon)}, \tag{A.1}$$

where, if the set in question is a bounded subset of a p-dimensional Euclidean space R^p, then $N(\varepsilon)$ is the minimum number of p-dimensional cubes of side ε needed to cover the set. For a point, a line and an area, $N(\varepsilon) = 1$, $N(\varepsilon) \sim \varepsilon^1$ and $N(\varepsilon) \sim \varepsilon^{-2}$, and eq. (A.1) yields $d_c = 0$, 1 and 2, as expected. However, for more

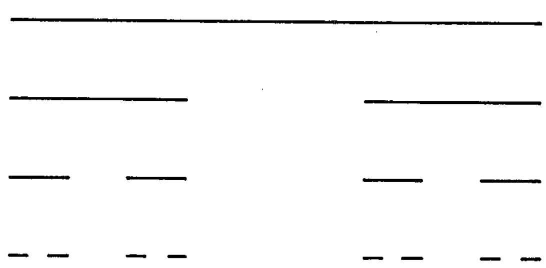

Fig. 1. – The first few steps in the construction of the classic example of a Cantor set.

general sets d_c can be noninteger. For example, consider the Cantor set obtained by the limiting process of deleting middle thirds, as illustrated in fig. 1. If we choose $\varepsilon = (1/3)^m$, then $N = 2^m$, and eq. (A.1) yields

$$d_c = \frac{\log 2}{\log 3} = 0.630\ldots. \tag{A.2}$$

Hausdorff dimension. – The capacity may be viewed as a simplified version of the Hausdorff dimension, originally introduced by HAUSDORFF in 1919 [9]. We have again reversed historical order and defined capacity before Hausdorff dimension because the definition of Hausdorff dimension is more involved.

To define the Hausdorff dimension of a set lying in a p-dimensional Euclidean space, consider a covering of it with p-dimensional cubes of variable edge length ε_i. Define the quantity $l_d(\varepsilon)$ by

$$l_d(\varepsilon) = \inf \sum_i \varepsilon_i^d, \tag{A.3}$$

where the infimum (*i.e.* minimum) extends over all possible coverings subject to the constraint that $\varepsilon_i \leqslant \varepsilon$. Now let

$$l_d = \lim_{\varepsilon \to 0} l_d(\varepsilon). \tag{A.4}$$

HAUSDORFF showed that there exists a critical value of d above which $l_d = 0$ and below which $l_d = \infty$. This critical value, $d = d_{\mathrm{H}}$, is the Hausdorff dimension. Precisely at $d = d_{\mathrm{H}}$, l_d may be either 0, ∞, or a positive finite number.

Weierstrass function. – In 1872 WEIERSTRASS [10] introduced the functions

$$K(x_1) = \sum_{n=0}^{\infty} \gamma^{-n(2-D)} \cos(\pi \gamma^n x_1), \tag{A.5}$$

and showed that they were nowhere differentiable in certain cases. HARDY [11] not only showed that $K(x_1)$ is nowhere differentiable for all $\gamma > 1$, and $0 < 2 - D \leqslant 1$, but, in addition, obtained some exact results concerning the local Lipschitz order of these functions. KAPLAN, MALLET-PARET and YORKE [12] show that the capacity of $K(x_1)$ is $d_c = D$ for $\gamma > 1$, $0 < 2 - D < 1$. MAULDIN and WILLIAMS [13] prove that the real part of the Weierstrass-Mandelbrot function $\mathrm{Re}\{W(x_1)\}$ given in (6.1) has Hausdorff dimension d_{H} bounded by

$$D - (C/\ln \gamma) \leqslant d_{\mathrm{H}} \leqslant D \tag{A.6}$$

for sufficiently large γ and some positive constant C.

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