THE MULTIFRACTAL STRUCTURE OF THE OCEAN SURFACE

by

MICHAEL STIASSNIE

in

NONLINEAR WATER WAVES WORKSHOP

University of Bristol, October 1991

Editor:
D.H. Peregrine

ISBN 0-86292-390-5
University of Bristol, Department of Mathematics

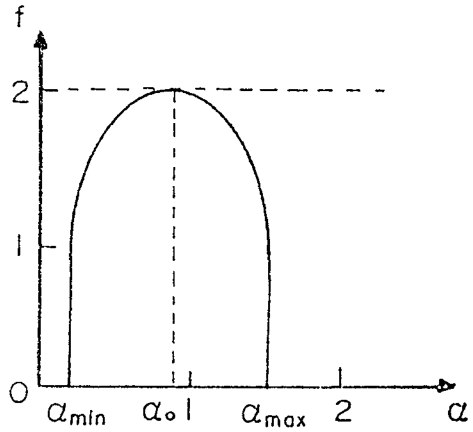
THE MULTIFRACTAL STRUCTURE OF THE OCEAN SURFACE

by

MICHAEL STIASSNIE*

The wavenumber spectrum of the ocean surface elevation contains detailed information regarding the large-scale structure of that surface. The high wavenumber behaviour of this spectrum contains some global information regarding small-scale structures and the singularities of the ocean surface. A more complete picture of these singularities is provided by the multifractal spectrum $f(\alpha)$.

The singularities of the surface are classified by their strength which is given by their Lipschitz-Holder exponent α. Each collection of singular points with a common $\alpha > 0$ forms a fractal set with dimension $f \leq 2$, embedded in the horizontal plane. A typical multifractal spectrum of a rough surface is shown in the figure.



The $f(\alpha)$ curve provides detailed information about the singular structure of the surface. For example, the fractal (box counting) dimension of the surface D, and the exponent β of the power-law decaying wavenumber spectrum (assuming small scale isotropy), expressed in terms of the $f(\alpha)$ curve properties are give by:

$$D = 1 - \alpha_1 + f(\alpha_1)\,, \text{ where } \frac{df}{d\alpha}\Big|_{\alpha=\alpha_1} = 1 \qquad (1)$$

$$\beta = 4 + 2\alpha_2 - f(\alpha_2)\,, \text{ where } \frac{df}{d\alpha}\Big|_{\alpha=\alpha_2} = 2 \qquad (2)$$

The derivation of (1) and (2) is given in the Appendix A.

Short gravity waves are probably isotropic, and it is well established that $\beta = 7/2$.

In order to obtain an approximate expression for the multifractal spectrum curve, we assume: i) that the ocean surface is regular almost everywhere, i.e. that $\alpha_0 = 1$, (see figure); and ii) that the $f(\alpha)$ curve (at least its more interesting left part) is not far off from a quarter of an ellipse.

---

*Coastal and Marine Engineering Research Institute, Department of Civil Engineering, Technion, Israel Institute of Technology, Haifa 32000, Israel

Equation (2) with $\beta = 7/2$ and the above assumptions yield

$$\left(\frac{f}{2}\right)^2 + \left(\frac{\alpha - 1}{0.75}\right)^2 = 1 \qquad (3)$$

The fractal dimension of the surface calculated from (1) and (3) is found to be $D = 2.13$. This is different from the D=2.25 which was obtained under the assumption of a monofractal surface (see Stiassnie, Agnon and Shemer, 1991).

Most of the small-scale energy is concentrated in singularites with Lipschitz-Holder exponent $\alpha_2 = 0.55$, and is spread over a fractal set with dimension $f_2 = 1.6$.

I suggest that the physical interpretation of the above result is that most of the deep ocean wave breaking process (whitecaps formation) is concentrated on a fractal set with dimension of about 1.6; a fact which could be checked by analyzing aerial photos of a stormy ocean surface. Note that f_2 is probably independent of the sea-state. The sea-state should be related to an appropriate measure of this set.

Note that for a case of a free water surface which is fractal the spatial derivatives do not exist, and the quadratic terms in the kinematic boundary condition become meaningless. To overcome this difficulty I propose, in Appendix B, a weak formulation of the water wave problem, which is derived from Hamilton's principle.

Appendix A: Multifractal Analysis of Rough Surfaces

1. *Definitions:*

Let z (x,y) be a continuous, bounded single-valued function in the unit square $0 \le x,y \le 1$. In the sequel we divide the unit square into a large number $N^2(\delta) = \delta^{-2}$ of indentical small squares with side δ. We also use discretized coordinate values $x_i = i\delta$, $y_j = j\delta$, i, j = 0,1,2,..., N.

The "range" of z (x,y) in a $\delta x \delta$ square is defined as

$$\Delta(x_i, y_j, \delta) = \sup z(x,y) - \inf z(x,y) \qquad (A.1)$$

where $x_i \le x \le x_i + \delta$ and $y_j \le y \le y_j + \delta$.

The box counting dimension of the surface z (x,y) is the value D for which the measure M has a finite (not zero and not infinite) value

$$M = \lim_{\delta \to 0} \delta^{D-1} \sum_{i,j=0}^{N-1} \Delta(x_i, y_j, \delta) \qquad (A.2)$$

The local behaviour of the function z (x,y) is represented by α - the Lipschitz-Holder (L-H) exponent defined by

$$\Delta (x_i, y_j, \delta) = g (x, y) \delta^{\alpha} \tag{A.3}$$

where g (x, y) is a good function (non-singular) of $x_i \leq x \leq x_i + \delta$ and of $y_i \leq y \leq y_i + \delta$.

The L-H exponent is always positive but may vary in a rather general manner. More specific, we denote by $f(\alpha) \leq 2$, the box-counting dimension of the set of points in the unit square which have a common α.

The number of δ x δ squares which is needed to cover the sets in the unit square for which z has L-H exponent in the range $(\alpha, \alpha + d\alpha)$ is

$$n (\alpha) = \rho_1 (\alpha) \delta^{-f(\alpha)} d\alpha \tag{A.4}$$

where ρ_1 is a good function. We also introduce the good function $\rho_2 (\alpha)$, as the average of g (x, y) in (A.3), taken over all points (x_i, y_j) for which the surface z (x, y) has L-H exponent α. The function f (α) is called the *Multifractal Spectrum.*

2. *Derivation of the f (α) Curve*

We start from a q - Measure

$$M_q = \lim_{\delta \to 0} \delta^{\tau(q)} \sum_{i,j=0}^{N-1} \mu_{ij}^q \tag{A.5}$$

where the relative weight of the i, j square is

$$\mu_{ij} (\delta) = \Delta (x_i, y_j, \delta) \Big/ \sum_{i,j=0}^{N-1} \Delta (x_i, y_j, \delta) \tag{A.6}$$

From (A.6), (A.3) and (A.2), for points (x_i, y_j) with L-H exponent α, we have

$$\mu (\alpha, \delta) = g (x, y) \delta^{\alpha} / M \delta^{1-D} \tag{A.6a}$$

The sigma (Σ) term on the r.h.s. of (A.5) is recalculated with the aid of (A.4) and (A.6a),

$$\sum_{i,j=1}^{N-1} \mu_{ij}^q = \int_0^{\infty} \rho_1 (\alpha) \delta^{-f(\alpha)} \left(\frac{\rho_2}{M}\right)^q \delta^{(D + \alpha - 1) q} d\alpha \tag{A.7}$$

Using the method of steepest descent to extract the dominant term from (A.7) in the limit of small δ

$$\lim_{\delta \to 0} \sum_{i,j=1}^{N-1} \mu_{ij}^q = \rho_1 (\alpha_q) \left(\frac{\rho_2 (\alpha_q)}{M}\right)^q \left(\frac{2\pi}{f'' (\alpha_q) \ell n \, \delta}\right)^{1/2} . \, \delta^{-f(\alpha_q) + q (D + \alpha_q - 1)} \tag{A.8}$$

where $f'(\alpha_q) = q$ and $f''(\alpha_q) < 0$ (A.8a)

Substituting (A.8) into (A.5), we get

$$\tau = f(\alpha) - q(D + \alpha - 1) \tag{A.9}$$

Taking the derivative with respect to q of (A.9) and applying (A.8a), we find

$$\frac{d\tau}{dq} = 1 - \alpha - D \tag{A.10}$$

Equations (A.9) and (A.10) enable to calculate f (α), once τ (q) is known.

3. *Fractal Dimension and Spectral Exponent*

From (A.2), (A.3) and (A.4)

$$M = \lim_{\delta \to 0} \delta^{D-1} \int_0^\infty \rho_1 \delta^{-f(\alpha)} \rho_2 \delta^\alpha \, d\alpha \tag{A.11}$$

Using the method of steepest descent and the fact that M is finite, gives

$$D = 1 - \alpha_1 + f(\alpha_1); \qquad \left.\frac{df}{d\alpha}\right|_{\alpha = \alpha_1} = 1 \tag{A.12}$$

In order to obtain the spectral exponent, we start by calculating an approximation to the mean-square increment assuming small-scale isotropy of the surface z

$$\left\langle [f(x + \delta\cos\theta, y + \delta\sin\theta) - f(x, y)]^2 \right\rangle \propto \int_0^\infty \frac{\delta^{-f(\alpha)}}{\delta^{-2}} \cdot (\delta^\alpha)^2 \, d\alpha \underset{\delta \to 0}{\propto} \delta^{-f(\alpha_2) + 2 + 2\alpha_2} \tag{A.13}$$

where $\left.\frac{df}{d\alpha}\right|_{\alpha = \alpha_2} = 2$

The high wavenumber limit of the spectrum is written as

$$\Psi(\underline{k}) \underset{k \to \infty}{\propto} k^{-\beta} \tag{A.14}$$

The Wiener-Khintchine relations enable us to compare the small increment limit (δ → 0) of the two-dimensional Fourier-transform of (A.14) with (A.13) which yields

$$\beta = 4 + 2\alpha_2 - f(\alpha_2) \tag{A.15}$$

Appendix B: Weak Formulation of the Water-Wave Problem

Hamilton's principle asserts that the motion of a dynamical system from one configuration to another renders stationary the integral

$$\iiint_R L \, dxdydt \tag{B.1}$$

where the appropriate Lagrangian L, as shown by Luke (1967), is given by the principle of stationary pressure

$$L = -\rho \int_{-\infty}^{\eta(\underline{x},t)} [\phi_t + 1/2(\nabla\phi)^2 + gz]dz, \tag{B.2}$$

R is an arbitrary region in the ($\underline{x}$,t) space, ρ is the density of the water. η is the free surface elevation and φ is the flow potential. Using standard procedure of calculus of variations and integrating by parts gives

$$-\hat{\delta}\iiint_R \frac{L}{\rho} dxdydt =$$

$$= \iiint_R \{(1/2(\nabla\phi)^2 + \phi_t + gz)_{z=\eta}\hat{\delta}\eta + \int_{-\infty}^{\eta} (\nabla\phi \bullet \nabla(\hat{\delta}\phi) + (\delta\phi)_t)dz\}dxdydt =$$

$$= \iiint_R \{ (\phi_t + 1/2(\nabla\phi)^2 + gz)_{z=\eta}\, \hat{\delta}\eta + [\frac{\partial}{\partial t}\int_{-\infty}^{\eta} \hat{\delta}\phi dz + \frac{\partial}{\partial x}\int_{-\infty}^{\eta} \phi_x\hat{\delta}\phi dz + \frac{\partial}{\partial y}\int_{-\infty}^{\eta} \phi_y\hat{\delta}\phi dz] -$$

$$- \int_{-\infty}^{\eta} (\phi_{xx} + \phi_{yy} + \phi_{zz})\, \hat{\delta}\phi dz - ((\eta_t - \phi_z + \eta_x\phi_x + \eta_y\phi_y)\, \hat{\delta}\phi)_{z=\eta} \}dxdydt = 0 \tag{B.3}$$

Here $\hat{\delta}(*)$ represents the variation of the quantity *. The term in square brackets integrates out to the boundaries of the domain R, and is equal to zero, since $\hat{\delta}\phi$ is assumed to vanish there. Setting $\hat{\delta}\eta = 0$ and $\hat{\delta}\phi = 0$ at $z = \eta$ yields

$$\iiint_R \int_{-\infty}^{\eta} (\phi_{xx} + \phi_{yy} + \phi_{zz})\hat{\delta}\phi dxdydzdt = 0 \tag{B.4}$$

for any appropriate $\hat{\delta\phi}$.

Since $\hat{\delta\eta}$ and $\hat{\delta\phi}$ at $z = \eta$ can take 'arbitrary' values, one obtains

$$\iiint_R [\phi_t + 1/2(\nabla\phi)^2 + gz]_{z=\eta}\hat{\delta\eta}\,dxdydt = 0, \tag{B.5}$$

$$\iiint_R [\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z)\hat{\delta\phi}]_{z=\eta}dxdydt = 0 \tag{B.6}$$

Replacing $\hat{\delta\eta}$ and all $\hat{\delta\phi}$ by Dirac's Delta function in (B.4) to (B.6) yields the classical Euler equations for water-waves. But, if one has fractal (weak) solutions in mind, one must be somewhat 'more careful'. Having in mind solutions with fractal properties, one could insert Dirac's delta function in (B.4) and (B.5); but not in (B.6) where $\hat{\delta\phi}$ is taken as a product of Dirac's delta function in time and an *appropriate spatial variation* denoted by $\hat{\delta\tilde{\phi}}$ (x,y)

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \qquad z < \eta, \tag{B.4'}$$

$$\phi_t + 1/2(\nabla\phi)^2 + g\eta = 0, \qquad z = \eta, \tag{B.5'}$$

$$\int_{-\infty}^{\infty}\!\!\int (\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z)_{z=\eta}\hat{\delta\tilde{\phi}}\,dxdy = 0, \tag{B.6'}$$

Again, for classical differentiable solutions one usually takes the following two-dimensional Dirac's Delta function

$\hat{\delta\tilde{\phi}} = \delta(\sqrt{(X-x)^2+(Y-y)^2})/2\pi\sqrt{(X-x)^2+(Y-y)^2}$, (see Courant and Hilbert 1962, p. 791).

The *Riesz Fractional Integral* in two dimensions of h(x,y) and of order ν, as defined by Riesz (1949) can serve as a generalization of Dirac's Delta function:

$$_{-\infty}I^{\nu}_{\infty}(f) = \frac{\Gamma(1-\frac{\nu}{2})}{\pi 2^{\nu}\Gamma(\frac{\nu}{2})} \int_{-\infty}^{\infty}\!\!\int h(x,y)[(X-x)^2+(Y-y)^2]^{\frac{\nu}{2}-1}\,dxdy, \tag{B.8}$$

According to Lighthill (1959) p. 28, $\lim_{\varepsilon\to 0}(\Gamma(\varepsilon))^{-1}|x|^{\varepsilon-1} \to 2\delta(x) = \delta(|x|)$,

thus

$$_{-\infty}I^{0}_{\infty}(f) = h(X,Y) \tag{B.9}$$

I suggest that equation (B.10), which was obtained from (B.7) with

$h = (\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z)\,|_{z=\eta}$, and $\nu = 1 - \alpha_{min}$ (α_{min} is showen in the figure) should be taken as a replacement to the so-called kinematic condition:

$$\int\!\!\int_{-\infty}^{\infty} \{\eta_t + \eta_x\phi_x + \eta_y\phi_y - \phi_z\}|_{z=\eta(x,y,t)}[(X-x)^2+(Y-y)^2]^{(1+\alpha_{min})/2} dxdy = 0, \qquad \text{(B.10)}$$

This completes the weak formulation and makes clear my opinion regarding the nature of the *appropriate* $\hat{\delta}\tilde{\phi}$

$$\hat{\delta}\tilde{\phi} = \frac{\Gamma(1-\frac{\nu}{2})}{\pi 2^{\nu}\Gamma(\frac{\nu}{2})}[(X-x)^2+(Y-y)^2]^{\frac{\nu}{2}-1} \qquad \text{(B.11)}$$

Note that one can show that although $\frac{dW}{dx}$ where W(x) is the univariate Weierstrass function of dimension 1+ν, does not exist, its fractional integral $_{-\infty}I^{\nu}_{\infty}\left(\frac{dW}{dx}\right)$ exists and is finite. The extension of this result to two dimensions guided me in the above formulation.

References

1. Courant, R. & Hilbert, D., 1962. Methods of Mathematical Physics, Vol. 2, John Wiley, 830p.
2. Lighthill, M.J., 1959. Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press, 79p.
3. Luke, J.C. 1967. A variational principle for a fluid with a free surface, J. Fluid Mech., 27, 395-397.
4. Riesz, M., 1949.L'integrale Riemann-Liouville et le probleme de Cauchy, Acta Mathematica, 81, 1-223.
5. Stiassnie, M., Agnon, Y., & Shemer, L. 1991. Fractal dimensions of random water surfaces, Physica D. 47, 341-352.

Acknowledgements

This study is supported by the U.S. Office of Naval Research under Grant No. N00014-91-J-1449.