INITIAL INSTABILITY AND LONG-TIME EVOLUTION OF STOKES WAVES

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ABSTRACT. The modified Zakharov equation is used to assess the long-time evolution of a system composed of a Stokes wave and two initially small disturbances.

The most important result is that a kind of Fermi-Pasta-Ulam recurrence phenomenon (which has already been reported for class I instabilities), exists also for class II instabilities.

1. INTRODUCTION

In a recent study (Stiassnie & Shemer, 1984) we derived a modified version of the Zakharov integral equation for surface gravity waves. This version includes higher order, class II, nonlinear interaction as well as the more familiar class I interaction. A linear stability analysis of the new equation was used to study some short-time aspects of class I and class II instabilities of a Stokes wave, yielding result in agreement with those of McLean (1982). It is our opinion that the present knowledge of the long-time evolution of class I is limited and of class II is almost nil.

Class I instability:
Wave flume experiments by Lake et al (1977) have shown how the disturbances grew in time, reached a maximum and then subsided. Furthermore, the experiments showed how the unsteady wave train became, at some stage of its evolution, nearly uniform again. Yuen & Lake (1982) used a numerical solution of the Zakharov equation to show that the evolution may be recurring (Fermi-Pasta-Ulam recurrence) or chaotic, depending on the choice of modes included in the calculation. Stiassnie & Kroszynski (1982) used the nonlinear Schrödinger equation to study analytically the evolution of a <u>three-wave system</u>, composed of a carrier and two initially small 'side-band' disturbances. Their recurrence period (given by a simple formula) is in good agreement with the numerical results.

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Class II instability:
To our knowledge, the only information available is that of the experiments by Su and Su et al. (1982). They have found that an initial two-dimensional wave train of large steepness evolved into a series of three-dimensional crescentic spilling breakers (class II), and was followed by a transition to a two-dimensional moduled wave train (class I). One can only speculate that the growth of the crescentic waves and their disappearance are one cycle of a recurring phenomenon. Note that any theoretical study of this process had to await the derivation of the modified Zakharov equation.

In the present paper we attempt to assess the long-time evolution of three-wave systems composed of a Stokes wave (also called carrier) and two most unstable, initially small disturbances.

The long-time evolution of class II as well as class I instabilities is considered for infinitely deep water. The theory is presented in par. 2 and the results in par. 3.

2. THEORY

The smallest number of wave trains required to enable significant non-linear interaction is three for class I as well as class II. In order that significant interactions will occur, these three waves have to form a nearly resonating 'quartet' for class I and a nearly resonating 'quintet' for class II. To form a 'quartet' or a 'quintet' out of three waves, one can 'count' one of the waves, say-a, twice for class I and three times for class II.

Linear stability analysis is enabled by assuming that the initial amplitudes of the two disturbances are much smaller than the amplitude of the carrier wave.

The wave numbers of the carrier (denoted by subscript a) and the disturbances (b and c) are

$$\underline{k}_a = k_o(1,0); \quad \underline{k}_b = k_o(1+p,q); \quad \underline{k}_c = (J-p,-q) \tag{2.1}$$

where:

$$J = \begin{cases} 1, & \text{for class I} \\ 2, & \text{for class II} \end{cases} \tag{2.2}$$

The regions of instability in the (p,q)-plane and the most-unstable disturbances (having the maximum growth rate) are discussed in Stiassnie and Shemer (1984).

The free surface elevation for the three-wave system is given by

$$\eta = \sum_{j=a,b,c} a_j \cos(\underline{k}_j \cdot \underline{x} - \int_o^t \Omega_j dt + \theta_j) \tag{2.3}$$

where $(x_1, x_2) = \underline{x}$ are the horizontal coordinates, t is the time and θ_j are the initial phase shifts. The wavenumbers $\underline{k}_j$ are given in (2.1) and the 'Stokes-corrected' frequencies Ω_j are given by:

$$\Omega_a = \omega_a + T_{aaaa}|R_a|^2 + 2T_{abab}|R_b|^2 + 2T_{acac}|R_c|^2 \tag{2.4a}$$

$$\Omega_b = \omega_b + 2T_{baba}|R_a|^2 + T_{bbbb}|R_b|^2 + 2T_{bcbc}|R_c|^2 \tag{2.4b}$$

$$\Omega_c = \omega_c + 2T_{caca}|R_a|^2 + 2T_{cbcb}|R_b|^2 + T_{cccc}|R_c|^2 \tag{2.4c}$$

where ω_j is related to $\underline{k}_j$ by the linear dispersion relation

$$\omega_j = (g|\underline{k}_j|)^{\frac{1}{2}} \quad \text{and} \quad R_j = \pi\left(\frac{2g}{\omega_j}\right)^{\frac{1}{2}} a_j e^{i\theta_j} \tag{2.5}$$

The governing equations for R_j are a discretized form of the Zakharov and modified Zakharov eqs. for class I and class II, respectively, given by:

$$\frac{dR_a}{dt} = -2iS_a^{(J)} \cdot (R_a^*)^J R_b R_c \exp i\left(\int_o^t \Omega_J dt\right) \tag{2.6a}$$

$$\frac{dR_b}{dt} = -iS_b^{(J)} \cdot R_c^*(R_a)^{J+1} \exp i\left(-\int_o^t \Omega_J dt\right) \tag{2.6b}$$

$$\frac{dR_c}{dt} = -iS_c^{(J)} \cdot R_b^*(R_a)^{J+1} \exp i\left(-\int_o^t \Omega_J dt\right) \tag{2.6c}$$

where $\Omega_J = (J+1)\Omega_a - \Omega_b - \Omega_c$ (2.7)
and

	$S_a^{(J)}$	$S_b^{(J)}$	$S_c^{(J)}$
J=1	T_{aabc}	T_{bcaa}	T_{cbaa}
J=2	$\frac{1}{2}(U^{(3)}_{aaabc}+U^{(3)}_{aaacb})$	$U^{(2)}_{bcaaa}$	$U^{(2)}_{cbaaa}$

the * denotes the complex conjugate, and the interaction coefficients T...., U..... are given in Stiassnie & Shemer (1984). Note that the present R_j is related to B_j of Stiassnie & Shemer through:

$$R_j = B_j \exp i\left(\int_o^t (\Omega_j - \omega_j)dt\right) \tag{2.8}$$

Applying the operation $R_j^* \cdot Eq(2.6j) + R_j \cdot Eq(2.6j)^*$ on each of the Equations (2.6j) j = a,b,c yields

$$\frac{d}{dt}|R_a|^2 = 4S_a^{(J)} \mathrm{Im}\{(R_a^*)^{J+1} R_b R_c \exp i(\int_o^t \Omega_J dt)\} \tag{2.9a}$$

$$\frac{d}{dt}|R_b|^2 = -2S_b^{(J)} \mathrm{Im}\{(R_a^*)^{J+1} R_b R_c \exp i(\int_o^t \Omega_J dt)\} \tag{2.9b}$$

$$\frac{d}{dt}|R_c|^2 = -2S_c^{(J)} \mathrm{Im}\{(R_a^*)^{J+1} R_b R_c \exp i(\int_o^t \Omega_J dt)\} \tag{2.9c}$$

A new real function Z is defined, so that

$$\frac{dZ}{dt} = \mathrm{Im}\{(R_a^*)^{J+1} R_b R_c \exp i(\int_o^t \Omega_J dt)\} \tag{2.10}$$

Substitution of (2.10) in (2.9) and integration yield:

$$|R_a|^2 = 4S_a^{(J)} Z + |r_a|^2 \tag{2.11a}$$

$$|R_b|^2 = -2S_b^{(J)} Z + |r_b|^2 \tag{2.11b}$$

$$|R_c|^2 = -2S_c^{(J)} Z + |r_c|^2 \tag{2.11c}$$

where $r_j = R_j(t=0)$ are the initial values. Using (2.6) one can show that

$$\frac{d}{dt} \mathrm{Re}\{(R_a^*)^{J+1} R_b R_c \exp i(\int_o^t \Omega_J dt)\} = -\Omega_J \frac{dZ}{dt}, \tag{2.12}$$

which, after integration, gives

$$Re\{(R_a^*)^{J+1}R_bR_c \exp i(\int_o^t \Omega_J dt)\} = -\int_o^Z \Omega_J dZ + Re\{(r_a^*)^{J+1}r_br_c\} \quad (2.13)$$

From (2.10) and (2.13) we obtain

$$(\frac{dZ}{dt})^2 = |R_a|^{2(J+1)}|R_b|^2|R_c|^2 - [-\int_o^Z \Omega_J dZ + Re\{(r_a^*)^{J+1}r_br_c\}]^2 \quad (2.14)$$

The r.h.s of (2.14), after substitution of (2.7), (2.4) and (2.11) is a known polynomial in Z of order (J+3), denoted by $P_{J+3}(Z)$.

The solution of (2.14) is

$$t = \int_o^Z dZ/\sqrt{P_{J+3}(Z)} \quad (2.15)$$

where Z is allowed to vary between two neighboring roots of the polynomial: Z=ZL and Z=ZR where ZL < 0, and ZR > 0.

From (2.15) it is clear that Z is periodic in time and that the recurrence period T is given by

$$T = 2\int_{ZL}^{ZR} dZ/\sqrt{P_{J+3}(Z)} \quad (2.16)$$

For class I we write $P_4(Z) = \sum_{\ell=0}^{4} a_\ell Z^{(4-\ell)}$. When $a_o > 0$ then the 4 roots are $Z_4 > Z_3 > 0 > Z_2 > Z_1$; giving Z_3=ZR and Z_2=ZL, and (2.15) has the explicit solution

$$Z = \frac{Z_4(Z_3-Z_2)sn^2(u,\kappa)-Z_3(Z_4-Z_2)}{(Z_3-Z_2)sn^2(u,\kappa)-(Z_4-Z_2)} \quad (2.17a)$$

where sn is the Jacobian elliptic function of argument u and modulus κ :

$$u = sn^{-1}(\beta,\kappa) - a_o^{1/2}\, t/\gamma \quad (2.17b)$$

$$\beta = \sqrt{(Z_4-Z_2)Z_3}\ /\ \sqrt{(Z_3-Z_2)Z_4} \quad (2.17c)$$

$$\gamma = 2/\sqrt{(Z_4-Z_2)(Z_3-Z_1)} \quad (2.17d)$$

$$\kappa = \sqrt{(Z_3-Z_2)(Z_4-Z_1)}/\sqrt{(Z_4-Z_2)(Z_3-Z_1)} \quad (2.17e)$$

The recurrence period for this case is given by

$$T = \frac{2}{a_o^{1/2}} K(\kappa) \quad (2.18)$$

where K is a complete elliptic integral. Expressions similar to (2.17) and (2.18) exist for $a_o < 0$. For class II, where the polynomial is of order five, we cannot express the solution in terms of tabulated functions, and we integrate (2.15) and (2.16) numerically. Once Z is found, we use (2.11) to obtain $|R_j|$, (2.5) to obtain a_j and (2.3) to obtain η, (note that $\int_o^t \Omega_j(t)dt = \int_o^Z \Omega_j(Z)\sqrt{P_{J+3}(Z)}\,dZ$).

3. RESULTS

3.1 Initial instability

For infinitely deep water, the most unstable disturbances have the following wave-numbers:

Class I: $\underline{k}_b = k_o(1+P_I,0)$, $\underline{k}_c = k_o(1-P_I,0)$

Class II: $\underline{k}_b = k_o(1.5,q_{II})$, $\underline{k}_c = k_o(1.5,-q_{II})$

Thus, the class I evolving wave field is two-dimensional, whereas the class II wave field is three-dimensional but symmetric.

The values of p_I and q_{II} as functions of the initial steepness of the Stokes wave (h/λ, h and λ are the wave height and wave-length, respectively) are given by the solid lines and full symbols in Fig. 1. The lines, for class I and class II were derived from the Zakharov equation and the modified Zakharov equation, respectively. The dots and squares are from McLean (1982), obtained by a numerical stability analysis of an exact finite amplitude Stokes wave. The 'dashed' lines and hollow symbols on the same figure, give the growth-rate of the most unstable disturbances. ($\mathrm{Im}\sigma$ in McLean, 1982). Here again, the lines are our results and the dots (for class I) and squares (for class II) are those of McLean. The agreement between the two sets of results is good for waves of small to moderate wave steepness, and less impressive for very steep waves. The most important result of this analysis is that for $h/\lambda > 0.1$, (from McLean, or $h/\lambda > 0.11$ from our calculation), the growth rate of the class II instability overtakes that of class I.

3.2 The recurrence period

The nondimensional recurrence period $\omega_a T$ as a function of the initial linear carrier steepness $a_o k_o$ ($a_o = a_a(t=0)$), is shown in Fig. 2 for three cases: (i) class I, $\theta = 0$, (ii) class I, $\theta = \pi/2$; (iii) class II, $\theta = \pi/2$. The phase-shift difference θ is given by $\theta_b + \theta_c - (J+1)\theta_a$ at t=0. For all three cases we chose the relative amplitude of the initial disturbance $\varepsilon_1 = a_b(t=0)/a_o = a_c(t=0)/a_o$ to be 0.1. Generally speaking, the recurrence period depends on three parameters: The carrier steepness $a_o k_o$, the relative amplitude of the initial disturbance ε_1, and the phase-shift difference θ. For class I Stiassnie & Kroszynsky (1982) obtained:

$$\omega_a T = \begin{cases} 2(a_o k_o)^{-2}[0.98 - 2\ln(\varepsilon_1) - \ln|\cos\theta|], & \theta \neq \pi/2 \\ 2(a_o k_o)^{-2}[1.67 - 4\ln(\varepsilon_1)], & \theta = \pi/2 \end{cases} \tag{3.1}$$

Eq. 3.1 is represented in Fig. 2 by the two lower dashed straight lines. These results, obtained from the nonlinear Schrödinger equation, are in fair agreement with the present class I claculations. The recurrence period for class II, ($\varepsilon_1 = 0.1$ and $\theta = 90°$) is given by the upper solid curve in Fig. 2. The dashed line below this curve has the slope 1:3, representing a relationship of the form $\omega_a T \alpha (a_o k_o)^{-3}$. The dependence of class II T on ε_1 and θ was found to be qualitatively similar to that of class I. Namely, the periods for $\varepsilon_1 = 0.01$ were found to be 1.65 to 2 times greater than those for $\varepsilon_1 = 0.1$, ($\ln(0.01)/\ln(0.1) = 2$); the largest period is obtained for $\theta = 90°$; and the smallest for $\theta = 0°$.

To obtain a better physical feeling, note that the recurrence periods for $k_o a_o = 0.36$ ($\theta = 90°$, $\varepsilon_1 = 0.1$) which are about equal for the two classes are 38 times the carrier period.

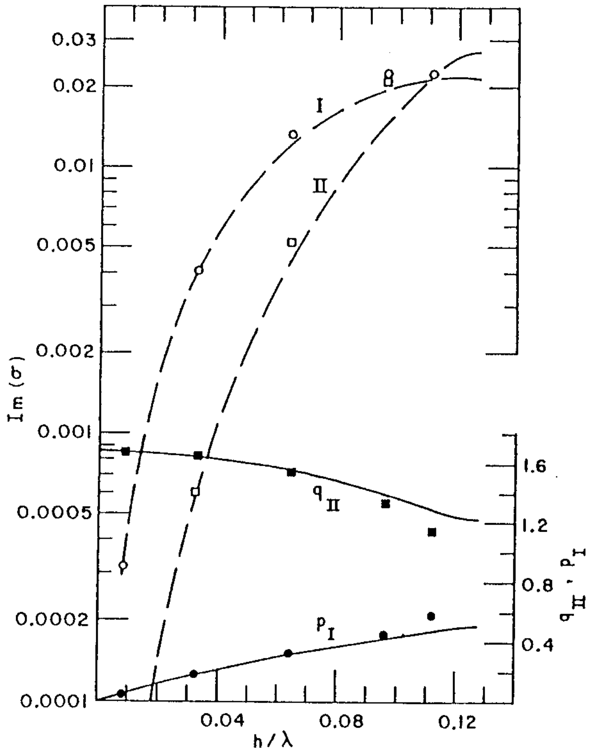

Fig. 1:The dependence of P_I, q_{II} and Im(σ) on h/λ

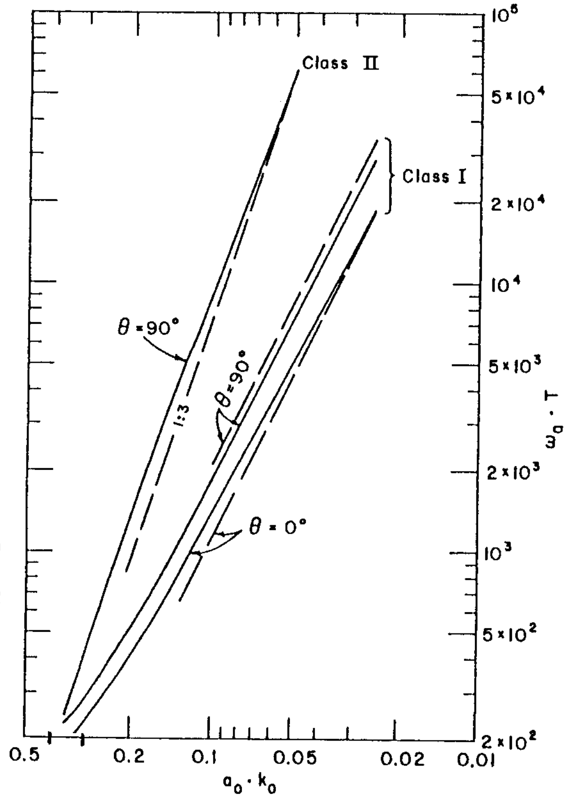

Fig. 2: Recurrence periods

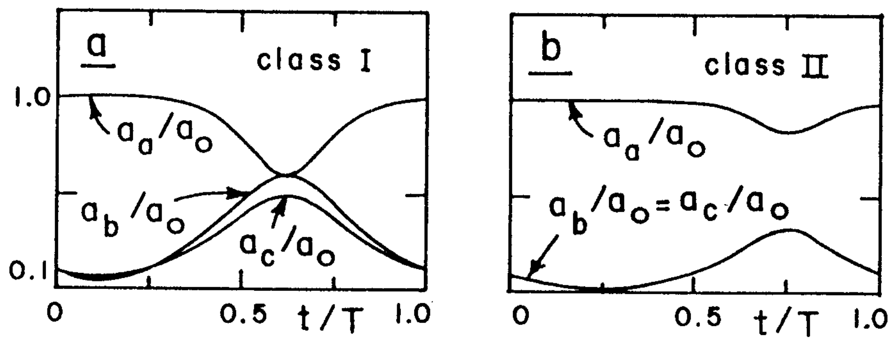

Fig. 3: Periodic amplitude evolution

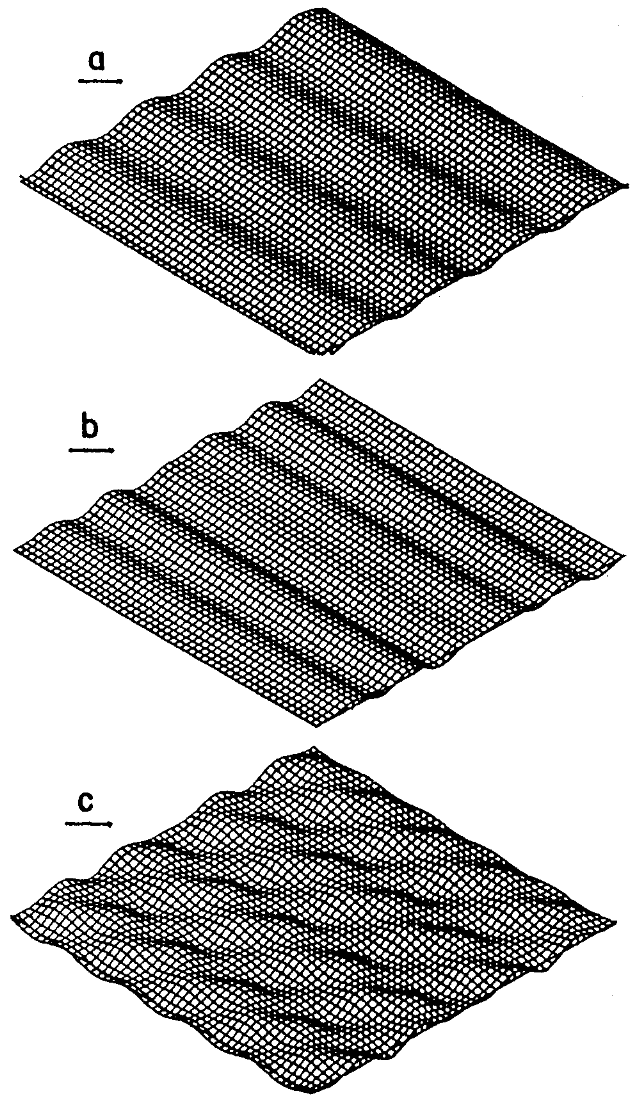

Fig. 4: Free surface elevation

3.3 Amplitude evolution

The evolution of the three amplitudes a_a/a_o, a_b/a_o, and a_c/a_o as a function of time t/T, (for $k_o a_o=0.29$, $\epsilon_1=0.1$, $\theta= 90°$) for class I and class II, is given in Fig. 3a and 3b respectively. Figures for the same $k_o a_o$ but different ϵ_1 are almost identical; and those with different $a_o k_o$, are rather similar. Both evolution processes have two distinct regions: (i) A region in which the disturbances stay smaller than their initial value. In this region the disturbances reach a minimum(at Z=ZR), which corresponds to an almost uniform wave train. (ii) A region in which the disturbances grow beyond their initial value. Here the disturbances reach a maximum (at Z=ZL),which corresponds to the most disturbed wave field. The order of appearance of these two regions depends on the sign of $\sin\theta$, see 2.9. Fig. 3 is typical for cases with $\sin\theta > 0$; For $\sin\theta < 0$ the order of the above- mentioned two regions is interchanged, so that the disturbances grow at the initial stage. Note that for class II $a_b=a_c$ throughout the evolution.

3.4 The free surface

Figure 4 shows the free surface elevation of $k_o a_o=0.36$, $\epsilon_1=0.1$ and $\theta = 90°$. The almost undisturbed wave-train is shown in Fig. 4a; the class I most modulated situation is given in Fig. 4b, and the crescentic shaped, class II waves in Fig. 4c.

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