On the application of fractional calculus for the formulation of viscoelastic models

Michael Stiassnie*

School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK
(Received 22 January 1979; revised 8 March 1979)

The concepts of fractional calculus, i.e. fractional integration and differentiation, are used for the formulation of a new constitutive equation which is expected to yield a representation of viscoelastic materials. The performance of the model, under a two-stage standard test, is theoretically investigated and a reasonable agreement with an experiment is obtained.

Introduction

Constitutive equations, giving the relations between the uni-directional stress σ and the respective strain ϵ, for an elastic material and a viscous material are as follows:

$$\sigma = k \cdot \epsilon(t) = k \cdot {}_0D_t^0 \epsilon(t) \tag{1}$$

$$\sigma = k \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = k \cdot {}_0D_t^1 \epsilon(t) \tag{2}$$

Equations (1) and (2) are the well known Hooke's law of elasticity and Newton's law of viscosity, respectively.†

The symbol ${}_0D_t^\alpha$ appearing in the right-hand-side of the above equations denotes differentiation of arbitrary order α with respect to t, using the notation recommended by Ross.[1]

Since the cases $\alpha = 0$ and $\alpha = 1$ correspond to purely elastic and purely viscous materials, respectively, it seems reasonable to assume that the equation:

$$\sigma = k \cdot {}_0D_t^\alpha \epsilon(t) \qquad (0 \leqslant \alpha \leqslant 1) \tag{3}$$

which is a straightforward generalization of equations (1) and (2), can represent the constitutive law for viscoelastic materials. According to the definition of the fractional derivative of order α, see Ross,[1] equation (3) is rewritten as:

$$\sigma(t) = k \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\xi)^{-\alpha} \epsilon(\xi)\, \mathrm{d}\xi \right] \tag{4}$$

where Γ is the gamma function.

On the other hand integrating equation (3) we obtain:

$$\epsilon(t) = k^{-1} \cdot {}_0D_t^{-\alpha} \sigma(t) = \frac{k^{-1}}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \sigma(\xi)\, \mathrm{d}\xi \tag{5}$$

where the symbol ${}_0D_t^{-\alpha}$ denotes integration of order α.

Hereditary integrals

After the integration by parts and the utilization of Leibnitz's rule for differentiation of integrals, equations (4) and (5) are converted into the standard form of hereditary integrals:

$$\sigma = \frac{k}{\Gamma(1-\alpha)} \left[\epsilon_0 t^{-\alpha} + \int_0^t (t-\xi)^{-\alpha} \frac{\mathrm{d}\epsilon}{\mathrm{d}\xi}\, \mathrm{d}\xi \right] \tag{6}$$

and:

$$\epsilon = \frac{k^{-1}}{\Gamma(1+\alpha)} \left[\sigma_0 t^{\alpha} + \int_0^t (t-\xi)^{\alpha} \frac{\mathrm{d}\sigma}{\mathrm{d}\xi}\, \mathrm{d}\xi \right] \tag{7}$$

respectively. Where $\epsilon_0 = \epsilon(t=0)$ and $\sigma_0 = \sigma(t=0)$.

Thus, it is seen that the relaxation modulus Y and the creep compliance J, have the following expressions:

$$Y = kt^{-\alpha}/\Gamma(1-\alpha) \tag{8}$$

$$J = k^{-1}t^{\alpha}/\Gamma(1+\alpha) \tag{9}$$

For $\alpha \to 0$, $Y \to k$ and $J \to k^{-1}$, as should be for an elastic material, see Flügge.[2]

When $\alpha \to 1$ it is easily seen that $J \to k^{-1} \cdot t$, as expected, while in order to establish that $Y \to k\delta(t)$ (δ is the delta

* On leave from Technion, Israel Institute of Technology, Haifa, Israel.

† All variables, in this article, are made dimensionless, with respect to some characteristic stress and time scales.

0307-904X/79/030300-03/$02.00

function) we refer to an identity given by Güttinger[3] (equation (4.29)).

Test of the model

In order to appraise the behaviour of a material modelled by equation (3) we perform a two-stage standard test. In the first stage we apply at $t = 0$ a constant stress $\sigma = \sigma_0$ and follow the variation in time of the strain $\epsilon(t)$. In the second stage, beginning at $t = t_1 > 0$, we fix the strain ϵ at whatever value ϵ_1 it reached, and follow the variations in time of the stress $\sigma(t)$.

Stage (i)

Substitutio. of $\sigma = \sigma_0$ and $d\sigma/dt = 0$ in equation (7) yields for $t \leqslant t_1$.

$$\epsilon/\epsilon_1 = (t/t_1)^\alpha; \quad \epsilon_1 = k^{-1}\sigma_0 t_1^\alpha/\Gamma(1+\alpha) \tag{10}$$

The variation of ϵ/ϵ_1 as a function of t/t_1 for five values of α ($\alpha = 0, 0.1, 0.5, 0.9, 1$) is shown in *Figure 1a.*

Stage (ii)

In order to calculate the stress for $t > t_1$, we substitute $\epsilon_0 = 0$, together with the following behaviour of $d\epsilon/dt$:

$$\frac{d\epsilon}{dt} = \begin{cases} \alpha k^{-1}\sigma_0 t^{\alpha-1}/\Gamma(1+\alpha) & t \leqslant t_1, \text{ see equation (10)} \\ 0 & t > t_1 \end{cases} \tag{11}$$

into equation (6), which gives:

$$\sigma/\sigma_0 = \frac{\alpha}{\Gamma(1-\alpha)\,\Gamma(1+\alpha)} \int_0^{t_1} (t-\xi)^{-\alpha}\xi^{\alpha-1}d\xi$$

$$= \frac{(t_1/t)^\alpha}{\Gamma(1-\alpha)\,\Gamma(1+\alpha)} F\left(\alpha, \alpha, 1+\alpha, \frac{t_1}{t}\right) \tag{12}$$

F is the hypergeometric function, see Abramowitz and Stegun.[4]

For $t = t_1$ the right-hand-side of equation (12) degenerates into one, i.e. $\sigma = \sigma_0$ as should be.

For the extreme values $\alpha \to 0, 1$ it can easily be shown that $\sigma/\sigma_0 \to 1, 0$ respectively, as expected.

For the specific case $\alpha = 0.5$ the hypergeometric function has a particularly simple expression. The stress relaxation for this case is given by:

$$\sigma/\sigma_0 = \frac{2}{\pi}\arcsin\,[(t_1/t)^{1/2}] \tag{13}$$

The behaviour of σ/σ_0 as a function of t/t_1 for $\alpha = 0, 0.1, 0.5, 0.9$ and 1 is represented in *Figure 1b.*

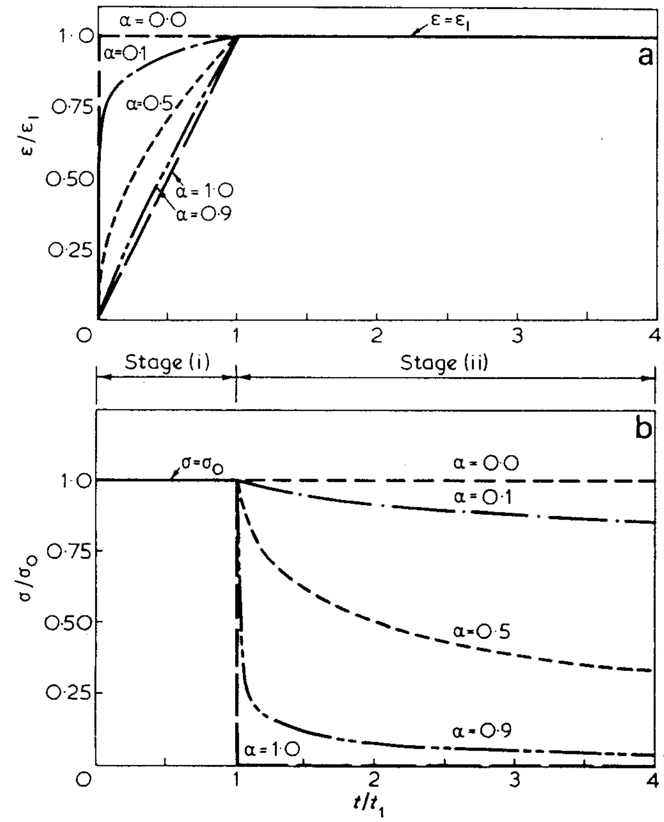

Figure 1 Standard test of model

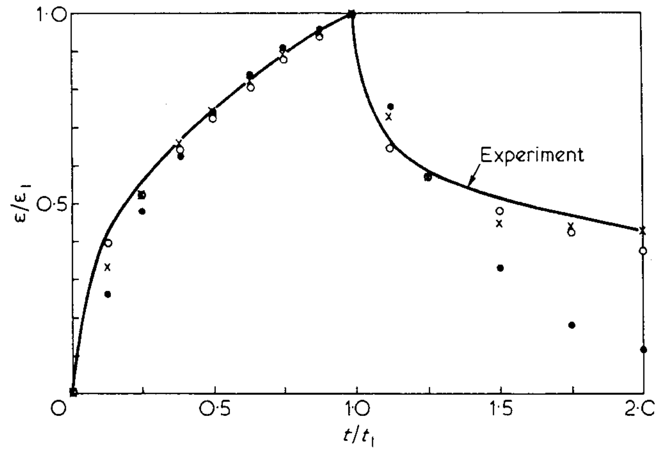

Figure 2 Creep and recovery in compression of nitrocellulose compound at 90°C. (o), present model; (•), KV element; (x), KV + dashpot

Comparison with experiment and other rheological models

The present model, which was developed in a rather formal way, certainly needs some 'real life' evidence in order to prove its value. In *Figure 2* we represent a comparison between an experiment of creep and recovery in compression of a nitrocellulose compound at 90°C which was taken from Bland[5] (p. 15), and the present model with $\alpha = 0.45$.

The agreement between model and experimental results is encouraging. It is important to note that the recovery stage in the above mentioned experiment is obtained after removing the constant load at a given instant, say $t = t_1$.

In order to model the loading process we have to substitute the expression $d\sigma/dt = -\sigma_0\delta(t - t_1)$ into equation (7) which results in:

$$\epsilon/\epsilon_1 = (t/t_1)^\alpha - (t/t_1 - 1)^\alpha \qquad t/t_1 > 1 \tag{14}$$

For $\alpha \to 0, 1$ the above expression gives $\epsilon/\epsilon_1 \to 0$ and 1, respectively.

Equations (10) and (14) have been used for the computation of the theoretical graph in *Figure 2.*

In order to compare the present model with conventional rheological models we show, in *Figure 2*, results obtained from fitting a Kelvin–Voigt element (which consists of a combination of a spring and dashpot in parallel) as well as a Kelvin–Voigt element in series with a dashpot to the experimental data. The appropriate nondimensional equations for the KV element (equation 15) and the KV element in series with a dashpot (equation 16) are:

$$\epsilon/\epsilon_1 = \begin{cases} \dfrac{1-\exp(-\beta_1 t/t_1)}{1-\exp(-\beta_1)} & t \leqslant t_1 \\ \dfrac{(\exp(\beta_1)-1)\exp(-\beta_1 t/t_1)}{1-\exp(-\beta_1)} & t > t_1 \end{cases} \quad (15)$$

$$\epsilon/\epsilon_1 = \begin{cases} \dfrac{1+\beta_0 t/t_1-\exp(-\beta_1 t/t_1)}{1+\beta_0-\exp(-\beta_1)} & t \leqslant t_1 \\ \dfrac{\beta_0+(\exp(\beta_1)-1)\exp(-\beta_1 t/t_1)}{1+\beta_0-\exp(-\beta_1)} & t > t_1 \end{cases} \quad (16)$$

The KV model, which is basically a one parameter model, similar to the present one (here $\beta_1 = 2.20$) is considerably less appropriate than the present model. The KV plus dashpot model gives an accuracy comparable to that of the present model but requires two fitting parameters ($\beta_0 = 0.71$ and $\beta_1 = 5.13$). No doubt that more generalized models of the above-mentioned type might improve the fitting but they lose the elegance and convenience of a one parameter model.

Addendum

After submission of this paper, I learnt that the rheologist Dr G. W. Scott-Blair used a similar approach about 30 years ago (for early references see Scott-Blair[6]).

The following quotations are part of a letter that I received from Dr Scott-Blair recently: 'I was working on the assessing of firmness of various materials (e.g. cheese and clay by experts handling them) these systems are of course both elastic and viscous but I felt sure that judgements were made not on an addition of elastic and viscous parts but on something in between the two so I introduced fractional differentials of strain with respect to time.' Later, in the same letter, Dr Scott-Blair adds: 'I gave up the work eventually, mainly because I could not find a definition of a fractional differential that would satisfy the mathematicians.'

References

1. Ross, B. In 'Fractional Calculus and its Applications' (Ed. A. Dold and B. Eckmann), Springer-Verlag, Berlin, 1974
2. Flügge, W. 'Viscoelasticity', Blaisdell Publishing Company, 1967
3. Güttinger, W. *Fortschr. Phys.* 1966, 14, 483
4. Abramowitz, M. and Stegun, I. A. 'Handbook of Mathematical Functions', N.B.S. Applied Math. Ser. 55, 1965
5. Bland, D. R. 'The Theory of Linear Viscoelasticity', Pergamon Press, Oxford, 1960
6. Scott-Blair, G. W. *J. Texture Stud.* 1974, 5, 3