IMA Journal of Applied Mathematics Advance Access published February 5, 2016

IMA Journal of Applied Mathematics (2016) Page 1 of 16
doi:10.1093/imamat/hxw003

Adapting Havelock's wave-maker theorem to acoustic-gravity waves

RAPHAEL STUHLMEIER* AND MICHAEL STIASSNIE
Faculty of Civil & Environmental Engineering, Technion – Israel Institute of Technology, 32000 Haifa, Israel
*Corresponding author: raphaels@tx.technion.ac.il

[Received on 23 March 2015; revised on 19 October 2015; accepted on 12 January 2016]

We investigate the different wave-modes generated by a wave-maker in compressible flow. In addition to the propagating and evanescent waves found in the incompressible case, new radially propagating acoustic-gravity modes appear. We discuss the asymptotic behaviour of these waves, and give an example for a simple line wave-maker configuration.

Keywords: acoustic-gravity waves; wave-makers; evanescent modes; compressible water waves.

1. Introduction

The problem of a wave-maker in infinitely deep water was first investigated by Havelock (1929). In subsequent years, Rhodes-Robinson (1971) and Hocking & Mahdmina (1991) extended Havelock's wave-maker theory to waves with surface tension, and considerable recent work has focused on extensions to different physical settings (e.g. Mohapatra *et al.*, 2011 for two-layer fluids, Chakrabarti & Sahoo, 1998 for porous wave-makers).

We study the problem of forced waves in water when compressibility is retained—this problem presents the novel phenomenon of acoustic-gravity modes, waves propagating radially outward at the speed of sound. Acoustic-gravity waves have received much recent interest, for example, in connection with tsunami (see Nosov, 1999 or Stiassnie, 2009) or deep-ocean circulation (see Kadri, 2014). While the generation mechanisms of acoustic-gravity waves have recently received attention also in terms of resonant wave interaction (see the discussion in Kadri, 2015), we intend to shed some light on their generation by wave-makers.

In contrast to earlier work on compressible flow, we focus on the case of infinite water depth. Some qualitative information about the behaviour of the different wave-components generated by a wave-maker is presented, in particular, the asymptotic behaviour of the new acoustic-gravity waves. We examine also the partition of energy into the different modes, and provide a numerical example for a line wave-maker.

In the bulk of this paper, we focus on waves where gravity is the sole restoring force, and as such will consider wave periods between 0.3 and 30 s (corresponding to frequencies ω between 0.2 and $20\,\mathrm{s}^{-1}$). Some remarks on higher frequencies, which necessitate the introduction of surface tension, are deferred to the discussion.

The paper is structured as follows: in Section 2, we formulate the governing equations for the wave-maker problem. In Section 3, we present the classical solution for the wave-maker in incompressible flow and the new, compressible solution. Section 4 is devoted to the physical interpretation of this new solution, and to the structure of the acoustic-gravity waves. The asymptotics presented therein are verified by energy considerations in Section 5, where the distribution of energy in the different modes is elucidated. Section 6 presents an explicit example, when the wave-maker is of δ-function type. Section 7 contains a comparison with the limiting cases of pure-acoustic and pure-gravity waves. Finally, Section 8 contains some concluding remarks and a discussion of some natural companion problems: that of finite water depth; the inclusion of surface tension for higher frequencies; and a brief discussion of the applicability of acoustic-gravity waves.

2. Formulation of the incompressible/compressible problems

We consider the 2D, irrotational motion of an inviscid fluid, whose velocity field may be described by a potential $\Phi(x,z,t)$, where x is the horizontal coordinate, z is measured vertically upward and t is time. The domain under consideration is $\{(x,z) \mid x \in \mathbb{R}^+,\ z \in (-\infty, 0]\}$, where the wave-maker is placed along the vertical line $x=0$. When compressibility is retained, the linearized field equation is the wave equation

$$\Phi_{tt} = c_{\mathrm{s}}^2 \left(\Phi_{xx} + \Phi_{zz}\right), \tag{2.1}$$

and c_{s} is the speed of sound. For an investigation of time-harmonic motions with frequency ω, this reduces our linear governing equations to the Helmholtz equation by substituting $\Phi = \mathrm{e}^{-\mathrm{i}\omega t}\phi$,

$$\phi_{xx} + \phi_{zz} + k_{\mathrm{s}}^2\phi = 0 \quad (k_{\mathrm{s}} = \omega/c_{\mathrm{s}}) \tag{2.2}$$

with the free-surface boundary condition (posed on the line $z=0$ after linearization)

$$\phi_z(x,0) = K\phi(x,0), \quad \left(K = \omega^2/g\right), \tag{2.3}$$

where g is the acceleration of gravity. We assume that there are no incoming waves from $+\infty$ in x and $-\infty$ in z (Sommerfeld's radiation condition). Neglecting the effects of compressibility is equivalent to allowing $c_{\mathrm{s}} \to \infty$, whereupon the field equation reduces to the Laplace equation. We further specify a Neumann boundary condition to formulate what is known as the wave-maker problem

$$\phi_x(0,z) = U(z), \tag{2.4}$$

for a wave-maker making periodic oscillations.

3. Solutions

3.1 *Havelock's solution for incompressible flow*

For the incompressible case, Havelock's solution to the wave-maker problem in infinite depth (Linton & McIver, 2001) is given by

$$\phi = \phi_{\mathrm{g}} + \phi_{\mathrm{e}} = 2a_0\,\mathrm{e}^{\mathrm{i}Kx}\,\mathrm{e}^{Kz} + \frac{2}{\pi}\int_0^\infty b(k)\,\mathrm{e}^{-kx}(k\cos(kz) + K\sin(kz))\,\mathrm{d}k, \tag{3.1}$$

where the coefficients are

$$a_0 = \frac{1}{i}\int_{-\infty}^0 U(z)\,\mathrm{e}^{Kz}\,\mathrm{d}z, \tag{3.2a}$$

$$b(k) = \frac{-1}{k}\int_{-\infty}^0 \frac{U(z)}{K^2+k^2}(k\cos(kz) + K\sin(kz))\,\mathrm{d}z. \tag{3.2b}$$

The solution contains progressive gravity waves (ϕ_{g}) and evanescent, standing waves (ϕ_{e}) which are only of local importance.

3.2 *The new solution for compressible flow*

For compressible flow, the solution takes the form:

$$\begin{aligned}\phi(x,z) = \phi_{\mathrm{g}} + \phi_{\mathrm{ag}} + \phi_{\mathrm{e}} = 2K\,\mathrm{e}^{Kz} a_0\, \mathrm{e}^{\mathrm{i}x\left(K^2+k_{\mathrm{s}}^2\right)^{1/2}} \\ + \frac{2}{\pi}\int_0^{k_{\mathrm{s}}} a(k)\,\mathrm{e}^{\mathrm{i}x\left(k_{\mathrm{s}}^2-k^2\right)^{1/2}}(k\cos(kz)+K\sin(kz))\,\mathrm{d}k \\ + \frac{2}{\pi}\int_{k_{\mathrm{s}}}^{\infty} b(k)\,\mathrm{e}^{-x\left(k^2-k_{\mathrm{s}}^2\right)^{1/2}}(k\cos(kz)+K\sin(kz))\,\mathrm{d}k.\end{aligned} \tag{3.3}$$

The coefficients $a_0,\ a,\ b$ are given as follows:

$$a_0 = \frac{-i}{\sqrt{K^2+k_{\mathrm{s}}^2}}\int_{-\infty}^{0} U(z)\,\mathrm{e}^{Kz}\,\mathrm{d}z, \tag{3.4a}$$

$$a(k) = \frac{-i}{\sqrt{k_{\mathrm{s}}^2-k^2}}\int_{-\infty}^{0} \frac{U(z)}{K^2+k^2}(k\cos kz + K\sin kz)\,\mathrm{d}z, \tag{3.4b}$$

$$b(k) = \frac{-1}{\sqrt{k^2-k_{\mathrm{s}}^2}}\int_{-\infty}^{0} \frac{U(z)}{K^2+k^2}(k\cos kz + K\sin kz)\,\mathrm{d}z. \tag{3.4c}$$

We may again identify the terms in (3.3) above with distinct physical regimes. Denoting these by

$$\phi = \phi_{\mathrm{g}} + \phi_{\mathrm{ag}} + \phi_e,$$

we find that they have the form of progressive gravity waves (ϕ_{g}), progressive acoustic-gravity waves (ϕ_{ag}) that exist on account of the compressibility (an analogous term is absent from (3.1)), and evanescent standing wave-modes (ϕ_{e}).

It is readily verified that the solution (3.3) satisfies (2.2–2.4), by making use of the identities (see Mei & Black, 1969)

$$\int_{-\infty}^{0} \psi(k,z)\psi(k',z)\,\mathrm{d}z = \frac{\pi\,\delta(k-k')}{2\sqrt{\left(k^2+K^2\right)\left((k')^2+K^2\right)}}, \tag{3.5a}$$

$$\int_{-\infty}^{0} \psi(k,z)\,\mathrm{e}^{Kz}\,\mathrm{d}z = 0, \tag{3.5b}$$

$$\text{for } \psi(k,z) = k\cos kz + K\sin kz, \tag{3.5c}$$

where δ is the Dirac delta function. In the limit as $k_{\mathrm{s}} \to 0$, Havelock's solution (3.1) is recovered from (3.3). Appendix A contains details of the derivation of this solution.

4. Physical interpretation of the solution for compressible flow

4.1 *Gravity mode*

The gravity mode in the compressible case

$$\phi_{\mathrm{g}} = 2K\,\mathrm{e}^{Kz} a_0\,\mathrm{e}^{\mathrm{i}x\left(K^2+k_{\mathrm{s}}^2\right)^{1/2}}$$

very much resembles that found in the incompressible problem. By considering $\mathfrak{Re}\{\phi_{\mathrm{g}}\,\mathrm{e}^{-\mathrm{i}\omega t}\}$, it is seen that its wave celerity

$$c = \sqrt{\frac{g}{K + g/c_{\mathrm{s}}^2}}$$

changes only slightly on account of the compressibility.

4.2 *Evanescent modes*

The term associated with evanescent modes is

$$\phi_{\mathrm{e}} = \frac{2}{\pi}\int_{k_{\mathrm{s}}}^{\infty} b(k)\,\mathrm{e}^{-x\left(k^2-k_{\mathrm{s}}^2\right)^{1/2}}(k\cos(kz) + K\sin(kz))\,\mathrm{d}k. \tag{4.1}$$

These standing waves exhibit decay and concurrent oscillation with depth z. As the waves considered have frequencies on the order of 2×10^{-1} to $2\times10^{1}\ \mathrm{s}^{-1}$ (or, equivalently, $1/3000 < k_{\mathrm{s}}/K < 1/30$), for practical considerations k_{s} can be neglected, and the evanescent modes behave essentially as those found in the incompressible case (3.1), which may be written as

$$\pi\phi_{\mathrm{e}} = \int_0^{\infty} b(k)\,\mathrm{e}^{-k(x-\mathrm{i}z)}(k-iK)\,\mathrm{d}k + \int_0^{\infty} b(k)\,\mathrm{e}^{-k(x+\mathrm{i}z)}(k+iK)\,\mathrm{d}k.$$

Repeated integration by parts then gives an asymptotic expansion in x and z

$$\pi\phi_e \sim b(0)K\frac{z}{x^2+z^2} + b(0)\frac{x^2-z^2}{\left(x^2+z^2\right)^2} + b_{\mathrm{s}}'(0)K\frac{2xz}{\left(x^2+z^2\right)^2} + \text{higher-order terms}. \tag{4.2}$$

We find $1/x^2$ polynomial decay with distance from the boundary, and oscillations and $1/z$ decay seen with depth. This polynomial decay with distance contrasts markedly with the exponential decay in x found for finite depth. In Section 6, this decay behaviour for the compressible case is depicted for a specified wave-maker condition.

4.3 *Acoustic-gravity modes*

We now turn our attention to the term

$$\phi_{\mathrm{ag}}(x,z) = \frac{2}{\pi}\int_0^{k_{\mathrm{s}}} a(k)\,\mathrm{e}^{\mathrm{i}x\left(k_{\mathrm{s}}^2-k^2\right)^{1/2}}(k\cos(kz) + K\sin(kz))\,\mathrm{d}k, \tag{4.3}$$

these waves being acoustic-gravity waves with a rightward propagating component.

By stationary phase methods (see Appendix B), we find that for large distances from the origin, the acoustic-gravity waves propagate radially outward throughout the fluid domain at the speed of sound c_{s}.

In polar coordinates $x = r\cos\alpha$ and $z = r\sin\alpha$, with $r \in \mathbb{R}^+,\ \alpha \in (-\pi/2, 0)$, we write the physical part of the potential for large r

$$\mathfrak{Re}\{\phi_{\mathrm{ag}}\,\mathrm{e}^{-\mathrm{i}\omega t}\}(r \gg 0) = \frac{\Gamma}{\sqrt{r}}\cos(k_{\mathrm{s}}r - \omega t + \gamma), \tag{4.4}$$

where

$$\Gamma = \sqrt{\frac{2}{k_{\mathrm{s}}\pi}} \cdot k_{\mathrm{s}}\cos(\alpha)\sqrt{K^2 + k_{\mathrm{s}}^2\sin^2(\alpha)} \cdot |a(-k_{\mathrm{s}}\sin(\alpha))|, \tag{4.5a}$$

$$\gamma = -\arctan\left(\frac{K}{k_{\mathrm{s}}\sin(\alpha)}\right) - \frac{\pi}{4} + \mathrm{Arg}(a(-k_{\mathrm{s}}\sin\alpha)). \tag{4.5b}$$

We see that the acoustic-gravity modes are radial progressive waves decaying as $1/\sqrt{r}$. In contrast to the finite-depth case (see Section 8.1, or Kadri & Stiassnie (2012) for further details) where at most finitely many acoustic-gravity modes exist, there is now a continuum of such modes with wavenumbers $k \in (0, k_{\mathrm{s}})$.

5. Verification by energy considerations

In what follows, we present some considerations of the energy flux for different modes in the wave-maker problem. In particular, we will use the distribution of energy to verify the asymptotic expansion for the acoustic-gravity modes. To this end, we revert to $\Phi = \phi \cdot \mathrm{e}^{-\mathrm{i}\omega t}$ and consider the real part of the resulting potential. According to Stoker (1992), the energy flux across a curve S is given by

$$F = \int_S p u_n \,\mathrm{d}S, \tag{5.1}$$

where $p = -\mathfrak{Re}\{\rho\partial\Phi/\partial t\}$ is the dynamic pressure, $u_n = \mathfrak{Re}\{\partial\Phi/\partial n\}$ is the velocity normal to S and ρ is the density of water. The time-averaged energy flux (denoted by a bar) across S is thus

$$\bar{F} = \frac{\omega}{2\pi}\int_S\int_0^{2\pi/\omega} p u_n \,\mathrm{d}t\,\mathrm{d}S. \tag{5.2}$$

Owing to the time averaging and relations (3.5a) and (3.5b) for $\psi(k, z)$ and e^{Kz}, it may be established that

$$\begin{aligned}\bar{F} &= -\frac{\rho\omega}{2\pi}\int_S\int_0^{2\pi/\omega}\left(\mathfrak{Re}\left\{\frac{\partial\Phi_{\mathrm{g}}}{\partial t}\right\}\mathfrak{Re}\left\{\frac{\partial\Phi_{\mathrm{g}}}{\partial n}\right\} + \mathfrak{Re}\left\{\frac{\partial\Phi_{\mathrm{ag}}}{\partial t}\right\}\mathfrak{Re}\left\{\frac{\partial\Phi_{\mathrm{ag}}}{\partial n}\right\}\right)\mathrm{d}t\,\mathrm{d}S \\ &= \bar{F}_{\mathrm{g}} + \bar{F}_{\mathrm{ag}}.\end{aligned} \tag{5.3}$$

In particular, we see that the time-averaged energy flux of the gravity modes Φ_{g} through vertical lines $x = x_0,\ x_0 \in (0, \infty)$

$$\bar{F}_{\mathrm{g}}(x_0, z) = \rho K\omega|a_0|^2\sqrt{K^2 + k_{\mathrm{s}}^2} \tag{5.4}$$

is independent of x_0, as is expected. For the time-averaged energy flux of the acoustic-gravity modes, we show that $\bar{F}_{\rm ag}$ through circular arcs $(r = r_0,\ -\pi/2 < \alpha < 0)$ is independent of the choice of r_0 for large r_0, and equal to the flux through the plane $x = 0$, denoted by $\bar{F}_{\rm ag}(x = 0)$.

Using (3.3) with (3.5a) and (3.5b), we find

$$\bar{F}_{\rm ag}(x=0) = \frac{\rho\omega}{\pi}\int_0^{k_{\rm s}} |a(\mu)|^2 \left(\mu^2 + K^2\right)\sqrt{k_{\rm s}^2 - \mu^2}\,{\rm d}\mu. \tag{5.5}$$

The time-averaged energy flux through a large circular arc is calculated using the asymptotic expansion (4.4) for $\Phi_{\rm ag}$

$$\bar{F}_{\rm ag}(r \gg 0) = \frac{\rho\omega}{\pi}k_{\rm s}^2\int_{-\pi/2}^{0} \cos^2(\alpha)\left(K^2 + k_{\rm s}^2\sin^2\alpha\right)|a(-k_{\rm s}\sin\alpha)|^2\,{\rm d}\alpha, \tag{5.6}$$

which coincides exactly with (5.5) upon changing variables to $\mu = -k_{\rm s}\sin\alpha$, and thus verifies the asymptotic form of $\Phi_{\rm ag}(r \gg 0)$.

6. A calculated example

While the previous sections give some general results on evanescent and acoustic-gravity modes, and the time-averaged energy fluxes, we can calculate these explicitly for the case of a line wave-maker of Dirac delta function type $\phi_x(0,\zeta) = U_0\delta(\zeta + h)$. Note that the units of U_0 are $\mathrm{m}^2/\mathrm{s}^1$. The coefficients of the potential are readily integrated, and (3.4a–3.4c) yield

$$a_0 = \frac{-iU_0}{\sqrt{K^2 + k_{\rm s}^2}}\,{\rm e}^{-Kh}, \tag{6.1a}$$

$$a(k) = \frac{U_0}{i\sqrt{-k^2 + k_{\rm s}^2}}\,\frac{k\cos kh - K\sin kh}{K^2 + k^2}, \tag{6.1b}$$

$$b(k) = \frac{-U_0}{\sqrt{k^2 - k_{\rm s}^2}}\,\frac{k\cos kh - K\sin kh}{K^2 + k^2}. \tag{6.1c}$$

The potential is

$$\Phi = \Phi_{\rm g} + \Phi_{\rm ag} + \Phi_e = \frac{2KU_0\,{\rm e}^{-Kh}}{i\sqrt{K^2 + k_{\rm s}^2}}\,{\rm e}^{Kz}\,{\rm e}^{{\rm i}\left(\sqrt{K^2+k_{\rm s}^2}x - \omega t\right)} \tag{6.2a}$$

$$+\frac{2U_0}{i\pi}\int_0^{k_{\rm s}} \frac{(k\cos kh - K\sin kh)(k\cos kz + K\sin kz)}{\left(K^2 + k^2\right)\sqrt{k_{\rm s}^2 - k^2}}\,{\rm e}^{{\rm i}\left(\sqrt{k_{\rm s}^2-k^2}x - \omega t\right)}\,{\rm d}k \tag{6.2b}$$

$$-\frac{2U_0}{\pi}\int_{k_{\rm s}}^{\infty} \frac{(k\cos kh - K\sin kh)(k\cos kz + K\sin kz)}{\left(K^2 + k^2\right)\sqrt{k^2 - k_{\rm s}^2}}\,{\rm e}^{-\sqrt{k^2-k_{\rm s}^2}x - {\rm i}\omega t}\,{\rm d}k. \tag{6.2c}$$

The behaviour of the potential amplitude of the evanescent modes for a choice of wave-maker depth $h = 4\pi$ m, can be seen in Fig. 1, in which (6.2c) was integrated numerically. The decay depicted here for the compressible case is indistinguishable from that in the incompressible case. Here and in all other examples $\omega = 1\ \mathrm{s}^{-1}$, $g = 9.81\ \mathrm{m/s}^2$, $U_0 = 1\ \mathrm{m}^2/\mathrm{s}$. This corresponds to a wavelength $\lambda \approx 60$ m, thus $h \approx \lambda/5$.

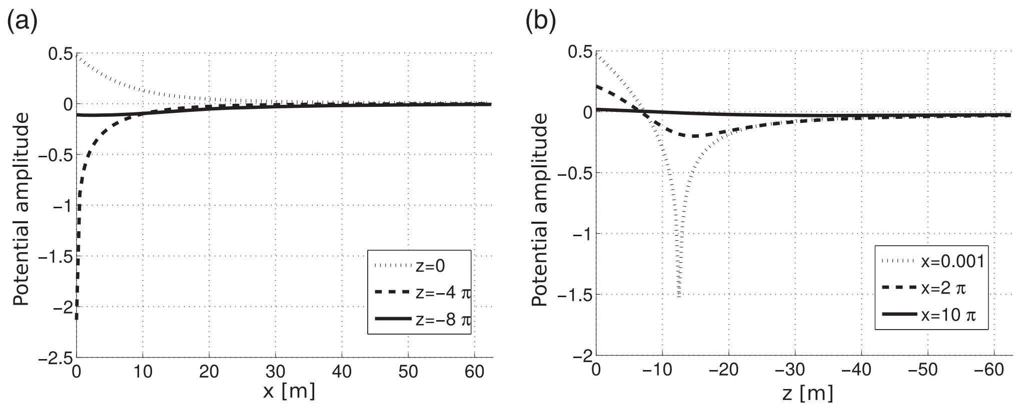

FIG. 1. Decay of the potential amplitudes (6.2c) of the evanescent modes for a δ-function wave-maker placed at a depth $h = 4\pi$ m, for $\omega = 1\,\mathrm{s}^{-1}$, $g = 9.81\,\mathrm{m/s}^2$ and $U_0 = 1\,\mathrm{m}^2/\mathrm{s}$. (a) Depicts the horizontal decay at different depths z, (b) the decay with depth at different positions x away from the wave-maker.

Within the framework of this explicit example, we can also elucidate the angular dependence of the amplitudes of the acoustic-gravity modes (4.4), retaining the notation for polar coordinates used in Section 4.3. The wave-fronts and contour lines of constant wave height (see (4.4), (4.5a)) are depicted in Fig. 2. For the frequencies under consideration ($0.2 < \omega < 20\,\mathrm{s}^{-1}$), the phase is taken to be independent of α, as $K/k_\mathrm{s} \gg 1$ and the argument of a is $\pi/2$, whence $\gamma = -\pi/4$. Thus, the wave-fronts are circular arcs. The wave height is given by

$$\frac{\Gamma}{\sqrt{r}} = \sqrt{\frac{2k_\mathrm{s}}{\pi r}}|\sin(\alpha)|U_0\left(\frac{1+Kh}{K}\right), \tag{6.3}$$

which may be seen to tend to zero as α tends to zero. Along each such curve (labelled for some sample depths by $k_\mathrm{s} z_i$ in Fig. 2), the value of the wave height is $\sqrt{2k_\mathrm{s}\pi^{-1}z_i^{-1}}U_0K^{-1}(1+Kh)$ (cf. (6.3)). Curves of constant wave amplitude are given by $r = z_i \sin^2\alpha$. We see that along the circular wave-fronts, the amplitude decreases as $\alpha \to 0$. Note that the asymptotic representation used to plot this figure is valid only for large distance from the origin, hence a neighbourhood of the origin must omitted from the figure.

The averaged power input from the wave-maker, as given by (5.2) along the wall (i.e. with $S = \{(x,z)\,|\,x=0,\ z\in(-\infty,0]\}$) may also be evaluated for the prescribed line wave-maker, and yields

$$\bar{F} = \bar{F}_g + \bar{F}_\mathrm{ag} = \frac{\rho\omega K U_0^2}{\sqrt{K^2+k_\mathrm{s}^2}}\,\mathrm{e}^{-2Kh} + \frac{\rho\omega U_0^2}{\pi}\int_0^{k_\mathrm{s}} \frac{(k\cos kh - K\sin kh)^2}{\left(K^2+k^2\right)\sqrt{k_\mathrm{s}^2-k^2}}\,\mathrm{d}k, \tag{6.4}$$

showing the distribution of the power in the gravity modes and the acoustic-gravity modes in the first and second term, respectively. The ratio $\bar{F}_\mathrm{ag}/\bar{F}_\mathrm{g}$ is visualized in Fig. 3, and demonstrates the growing importance of the acoustic-gravity modes compared with gravity modes with increasing depth of the wave-maker. For higher frequencies, the relative importance of the acoustic-gravity modes with depth increases faster still.

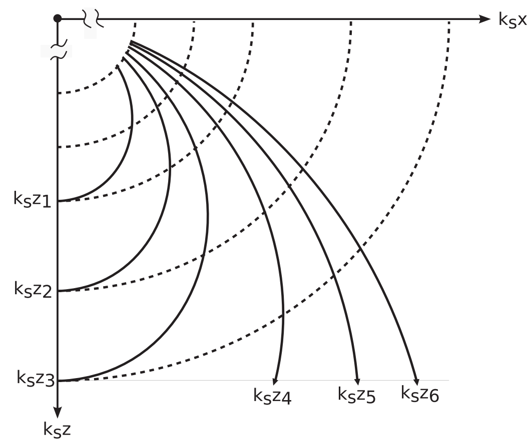

FIG. 2. Schematic depiction of contour lines of constant wave height $\Gamma/r_i^{1/2}$ (solid curves) for different $k_s z_i \gg 1$, and the circular arcs of the wave-fronts (dashed lines) of the acoustic-gravity waves. The domain considered (outside the innermost dotted arc) is such that the asymptotic representation (4.4) is valid.

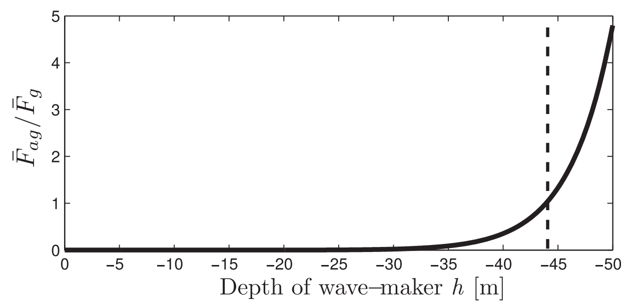

FIG. 3. Graph of the ratio $\bar{F}_{ag}/\bar{F}_g$ with depth h of the wave-maker, for $\omega = 1\,\mathrm{s}^{-1}$, $g = 9.81\,\mathrm{m/s}^2$ and depth varying from 0 to 50. $\bar{F}_{ag}/\bar{F}_g = 1$ for $h = 43.9$ m (dotted line).

The integral expression for $\bar{F}_{ag}$ may be evaluated approximately (assuming $k \leqslant k_s \ll K$) to yield

$$\bar{F}_{ag} \approx \frac{\rho U_0^2 g}{4\omega c_s^2}(1 - Kh)^2, \tag{6.5}$$

which coincides with the time-averaged power of the acoustic-gravity modes ϕ_{ag} through a quarter-circle with large radius r, by the considerations of the previous section.

7. Comparison with limiting cases $c_s \to \infty$ and $g \to 0$

As we have seen from the structure of the solution in Section 4 and the explicit calculations in Section 6, the relative importance of the acoustic-gravity and surface-gravity components changes with depth. This leads one to consider the relation between the physical scenario presented here, combining the effects of gravity and compressibility, and two natural limiting problems that arise from our formulation: the well-known *incompressible problem* where the elasticity of the fluid is neglected ($c_s \to \infty$) leading to a change of the field equation (2.2) to

$$\Delta\phi = 0, \tag{7.1}$$

with the solution given in (3.1–3.2b).

The other limit is the *pure-acoustic problem* where gravity is neglected ($g \to 0$, hence $K \to \infty$) and the surface condition (2.3) becomes

$$\phi = 0 \quad \text{on } z = 0. \tag{7.2}$$

(See Jensen *et al.*, 2011 for the formulation of this canonical boundary value problem in ocean-acoustics). For the pure-acoustic case, separation of variables again allows a treatment of the problem. In this case, the completeness relation (cf. Appendix A) for the z-separated problem is simply the Fourier sine transform, which yields a solution

$$\phi(x,z) = \frac{2}{\pi}\int_0^{k_s} a(k)\,\mathrm{e}^{\mathrm{i}x\left(k_s^2-k^2\right)^{1/2}}\sin(kz)\,\mathrm{d}k + \frac{2}{\pi}\int_{k_s}^{\infty} b(k)\,\mathrm{e}^{-x\left(k^2-k_s^2\right)^{1/2}}\sin(kz)\,\mathrm{d}k, \tag{7.3}$$

with

$$a(k) = \frac{-i}{\sqrt{k_s^2-k^2}}\int_{-\infty}^{0} U(z)\sin(k\zeta)\,\mathrm{d}\zeta, \tag{7.4}$$

$$b(k) = \frac{-1}{\sqrt{k^2-k_s^2}}\int_{-\infty}^{0} U(z)\sin(k\zeta)\,\mathrm{d}\zeta. \tag{7.5}$$

The lack of surface-gravity waves excepted, we find an unsurprising resemblance to the evanescent ($b(k)$) and propagating ($a(k)$) terms found in the acoustic-gravity setting (cf. (3.3)). An analysis of the asymptotic behaviour of these progressive acoustic-only modes along the lines of (4.4) reveals that the asymptotics are qualitatively identical. A numerical comparison of potential amplitudes for the two types of propagating acoustic waves (with and without gravity) demonstrates that, as the depth of the wave-maker increases and the relative importance of the gravity components declines, so too do the acoustic-gravity and pure-acoustic waves grow to resemble one another.

Figure 4 compares the three situations in terms of the time-averaged energy flux, i.e. the total power input to the wave-maker. The solid curve represents the total power input to a wave-maker placed at depth h for the problem with both compressibility and gravity effects, and shows the sum $\bar{F} = \bar{F}_g + \bar{F}_{\mathrm{ag}}$ in (6.4). The dashed line depicts the time-averaged energy flux for the propagating gravity modes ϕ_{g} of (3.1). Finally, the dotted line depicts the same for pure-acoustic modes (7.3).

It may be observed that the pure-gravity and acoustic-gravity problems come to resemble one another as the depth of the wave-maker decreases, and begin to diverge for greater wave-maker depths, as the relative importance of the gravity modes wanes and the acoustic components of the combined acoustic-gravity solution become predominant (see also Fig. 3 for a direct comparison). With increasing depth, the pure-acoustic and acoustic-gravity problem converge. As the depth of the wave-maker $h \to \infty$, $\bar{F}_a$ and $\bar{F}_{g+a}$ approach one another asymptotically.

8. Discussion

We have presented a treatment of the classical wave-maker problem, taking into account the effects of compressibility, and putting particular emphasis on understanding the acoustic-gravity modes arising therein. These results may be brought to bear on general problems of wave-structure interaction, where

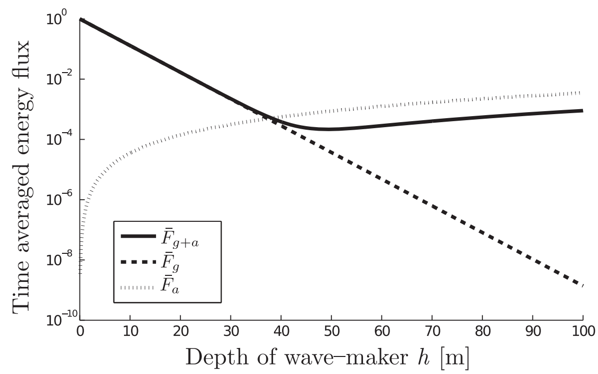

FIG. 4. Comparison of time-averaged energy fluxes for pure-gravity, pure-acoustic and acoustic + gravity cases, and δ-function wave-makers, plotted vs. depth h of the wave-maker, for $\omega = 1\ \mathrm{s}^{-1}$, $g = 9.81\ \mathrm{m/s^2}$, $U_0 = 1\ \mathrm{m^2/s}$, $c_s = 1500\ \mathrm{m/s}$ and depth varying from 0 to 100.

the asymptotics and behaviour described above may be considered prototypical for certain 2D configurations. In the balance of this section, we compare our results with the better known finite depth solution, and more importantly discuss the extension of our new solution to higher frequencies, which requires the introduction of surface tension. We subsequently make some brief comments on applications of acoustic-gravity waves.

8.1 *Finite depth*

The investigation of acoustic-gravity waves in infinite depth, presented here for the first time, may be compared with the better known finite depth case. There, the boundary value problem (2.2), (2.3) is supplemented by the bottom boundary condition

$$\phi_z = 0 \quad \text{on the bed } z = -H. \tag{8.1}$$

The resulting regular Sturm–Liouville problem for the z-variable leads to the well-known expansion of the potential in terms of eigenfunctions (which may be contrasted with the situation in infinite depth as presented in Appendix A):

$$f_0(z) = \sqrt{2}\cosh(k_0(z+H)) * \left(H + K^{-1}\sinh^2(k_0 H)\right)^{-1/2},$$
$$f_n(z) = \sqrt{2}\cos(k_n(z+H)) * \left(H - K^{-1}\sin^2(k_n H)\right)^{-1/2},$$

corresponding to the eigenvalues $k_i,\ i = 0, 1, 2, \ldots$ given by

$$k_0 \tanh(k_0 h) = K, \quad k_n \tan(k_n h) = -K$$

(see Mei *et al.*, 2005). The coefficients in the eigenfunction expansion

$$\phi(x,z) = a_0(x) f_0(z) + \sum_{n=1}^{\infty} a_n(x) f_n(z) \tag{8.2}$$

are obtained from the Helmholtz equation, which yields right- and left-going progressive waves

$$a_0(x) = \alpha_0\, \mathrm{e}^{\mathrm{i}\left(k_0^2+k_{\mathrm{s}}^2\right)^{1/2}x} + \beta_0\, \mathrm{e}^{-\mathrm{i}\left(k_0^2+k_{\mathrm{s}}^2\right)^{1/2}x}, \tag{8.3}$$

as well as ostensibly 'evanescent' modes

$$a_n(x) = \alpha_n\, \mathrm{e}^{\left(k_n^2-k_{\mathrm{s}}^2\right)^{1/2}x} + \beta_n\, \mathrm{e}^{-\left(k_n^2-k_{\mathrm{s}}^2\right)^{1/2}x}. \tag{8.4}$$

It is in these latter that the effects of compressibility come to bear most significantly—the eigenvalues $k_n \in \mathbb{R}^+$ may be ordered as a monotonically increasing series, with $k_n h \in ((n-1/2)\pi, n\pi)$, so that for a given fluid depth h and frequency ω there may be finitely many acoustic-gravity waves corresponding to

$$k_n^2 - k_{\mathrm{s}}^2 < 0. \tag{8.5}$$

If the depth h is sufficiently large to allow such modes, the potential of the N (right-going) acoustic-gravity waves may be written

$$\Phi^+_{\mathrm{ag}}(x,z) = \sum_{n=1}^{N} \frac{\sqrt{2}\alpha_n}{\left(h - K^{-1}\sin^2(k_n h)\right)^{1/2}} \cos\left(\sqrt{k_{\mathrm{s}}^2 - k_n^2}x - \omega t\right)\cos\left(k_n(z+h)\right). \tag{8.6}$$

Each of these acoustic-gravity modes may be rewritten as a sum of two waves, whose wavevector has magnitude k_{s}, and which propagate at an angle

$$\theta = \pm \arctan\left(\frac{k_n}{\sqrt{k_{\mathrm{s}}^2 - k_n^2}}\right) \tag{8.7}$$

to the horizontal x.

This finite number of acoustic-gravity modes in finite depth contrasts markedly with the continuum of such modes, for wavenumbers k between 0 and k_{s}, found in the case of infinite water depth. Moreover, while in finite depth the energy of the acoustic-gravity waves is trapped in the 'duct' between surface and bed, the infinite depth acoustic-gravity modes propagate only radially outward and downward into the fluid domain; for large distances from the wave-maker their amplitudes decay as $r^{-1/2}$.

8.2 *Effects of surface tension*

While we have here focused on gravity waves ($0.2 < \omega < 20\,\mathrm{s}^{-1}$), for higher frequencies—leading to waves with a wavelength much <10 cm—the effects of surface tension are expected to become increasingly important. The main change comes from adding to the surface dynamic boundary condition a term dependent on the surface curvature, so that it takes the form

$$\rho\Phi_t + \rho g\eta - T\eta_{xx} = 0 \quad \text{on } z = 0,$$

where T is the surface tension (a reference value for water at 20°C is $T = 0.073$ N/m, Vargaftik *et al.*, 1983). Combining this with the kinematic boundary condition $\eta_t = \Phi_z$, and the wave equation (2.1) leads to

$$\Phi_{tt} + g\Phi_z - \frac{T}{\rho}\left(\frac{\Phi_{ttz}}{c_s^2} - \Phi_{zzz}\right) = 0 \quad \text{on } z = 0.$$

The resulting boundary value problem for the periodic line wave-maker, written in terms of ϕ, is stated as

$$\phi_{xx} + \phi_{zz} + k_s^2\phi = 0 \quad \text{for } z \leqslant 0, x > 0, \tag{8.8}$$

$$\phi_x(0,z) = U_0\delta(z-h) \quad \text{for } h \in (-\infty, 0), \tag{8.9}$$

$$\phi_z + M\phi_{zzz} = K\phi \quad \text{on } z = 0, \tag{8.10}$$

where we neglect a small correction $Mk_s^2 \ll 1$ to the term ϕ_z in (8.10). Here $K = \omega^2/g$, $M = T/\rho g$ and the dispersion relation with surface tension becomes

$$K = \kappa\left(1 + M\kappa^2\right). \tag{8.11}$$

This corresponds to the problem of forced waves with surface tension in incompressible water as treated by Rhodes-Robinson (1971).[1]

The solution to (8.8–8.10) is given by

$$\phi(x,z) = 2\left(1 + M\kappa^2\right) a_0\, e^{\kappa z}\, e^{ix\sqrt{\kappa^2+k_s^2}} + \frac{2}{\pi}\int_0^{k_s} a(k)\, e^{ix\sqrt{k_s^2-k^2}}\, \tilde{\psi}(k,z)\, dk$$
$$+ \frac{2}{\pi}\int_{k_s}^{\infty} b(k)\, e^{-x\sqrt{k^2-k_s^2}}\, \tilde{\psi}(k,z)\, dk$$

for $\tilde{\psi}(k,z) = (k(1 - Mk^2)\cos kz + K\sin kz)$. The coefficients are:

$$a_0 = \frac{-i\kappa U_0\, e^{\kappa h}}{\left(1 + 3M\kappa^2\right)\sqrt{\kappa^2 + k_s^2}}, \tag{8.12a}$$

$$a(k) = \frac{-iU_0}{\sqrt{k_s^2 - k^2}} \cdot \frac{\tilde{\psi}(k,-h)}{k^2\left(1 - Mk^2\right)^2 + K^2}, \tag{8.12b}$$

$$b(k) = \frac{-U_0}{\sqrt{k^2 - k_s^2}} \cdot \frac{\tilde{\psi}(k,-h)}{k^2\left(1 - Mk^2\right)^2 + K^2}. \tag{8.12c}$$

The verification of (8.8) and (8.10) is immediate, and the verification of the wave-maker condition (8.9) is identical to that presented in Rhodes-Robinson (1971). Indeed, for $c_s \to \infty$, the solution of Rhodes-Robinson is recovered, and for $T \to 0$ we find that of Section 6.

[1] In Rhodes-Robinson, the boundary value problem is supplemented by a condition specifying the free-surface slope at the wave-maker. We consider the case of a submerged line wave-maker, and assume the free-surface slope at the boundary $x = 0$ is zero.

If we consider waves with frequencies $20 < \omega < 2000\,\mathrm{s}^{-1}$, we find very good numerical agreement between the evanescent modes with and without surface tension. The asymptotics for the acoustic-gravity modes (4.4) are identical save for a factor $Mk_s^2 \sin^2\alpha$ that appears in

$$\Gamma = \sqrt{\frac{2}{k_s\pi}} k_s \cos(\alpha) \sqrt{K^2 + \left(k_s \sin\alpha \left(1 - Mk_s^2 \sin^2\alpha\right)\right)^2}, \tag{8.13a}$$

$$\gamma = -\arctan\left(\frac{K}{k_s \sin\alpha \left(1 - Mk_s^2 \sin^2\alpha\right)}\right) - \frac{\pi}{4} + \mathrm{Arg}(a(-k_s \sin\alpha)) \tag{8.13b}$$

(cf. (4.5a), (4.5b)). This factor may safely be neglected, as $Mk_s^2 \ll 1$ for the above-mentioned physical values of ω.

8.3 *Applications of acoustic-gravity theory*

Having presented the linear wave-maker theory in infinite depth without the usual neglect of the compressibility of water, one may ask what regions of applicability exist for these new waves. The contributions of acoustic-gravity waves to generating microseisms in the sea-bed were first investigated in the mid-20th century by Longuet-Higgins (1950), and their role in generating low-frequency oceanic noise has been the focus of considerable recent study (see e.g. Ardhuin *et al.*, 2013 which also compares with measurements, and the discussion in Kadri, 2015). The connection between tsunamigenic earthquakes and acoustic-gravity waves generated in connection with tsunami dates back at least to Yamamoto (1982), and in recent years some direct measurements of these compressibility waves have been made (see, e.g. Nosov *et al.*, 2007). The idea of using such earthquake-generated acoustic-gravity waves to improve tsunami warning systems has also been the focus of much recent work (Stiassnie, 2009; Eyov *et al.*, 2013).

Just as Havelock's wave-maker theorem in incompressible flow provides a basis for the analysis of problems in wave–structure interaction (Ursell, 1947, 1948), so it is to be hoped that the utility of our expansion in compressible flow will extend beyond the simple example of the δ-function wave-maker we have used as a numerical example. In particular, we have shown the evident but otherwise not quantitatively assessable fact that some of the energy in any instance of wave–structure interaction will go into the generation of acoustic-gravity waves, in a depth- and frequency-dependent manner. The relationship between compressibility and gravity, and the relative importance of each in different physical settings which we have touched upon in Section 7 provide interesting avenues for future research.

Funding

This research was supported by the Israel Science Foundation (grant 464/13). R.S. was also supported in part at the Technion by a fellowship from the Lady Davis Foundation.

References

Ardhuin, F., Lavanant, T., Obrebski, M., Marié, L., Royer, J.-Y., D'Eu, J.-F., Howe, B. M., Lukas, R. & Aucan, J. (2013) A numerical model for ocean ultra-low-frequency noise: wave-generated acoustic-gravity and Rayleigh modes. *J. Acoust. Soc. Am.*, **134**, 3242–3259.

Chakrabarti, A. & Sahoo, T. (1998) The effect of surface tension in porous wave maker problems. *J. Aust. Math. Soc. Ser. B. Appl. Math.*, **39**, 539–556.

Eyov, E., Klar, A., Kadri, U. & Stiassnie, M. (2013) Progressive waves in a compressible-ocean with an elastic bottom. *Wave Motion*, **50**, 929–939.

Friedman, B. (1990) *Principles and Techniques of Applied Mathematics,* 1st edn. Dover Publications, 315 pp.

Havelock, T. (1929) Forced surface-waves on water. *London, Edinburgh, Dublin Philos. Mag. J. Sci.*, **8**, 569–576.

Hocking, L. M. & Mahdmina, D. (1991) Capillary-gravity waves produced by a wavemaker. *J. Fluid Mech.*, **224**, 217–226.

Jensen, F. B., Kuperman, W. A., Porter, M. B. & Schmidt, H. (2011) *Computational Ocean Acoustics*. New York, NY: Springer.

Kadri, U. (2014) Deep ocean water transport by acoustic-gravity waves. *J. Geophys. Res. Ocean.*, **119**, 7925–7930.

Kadri, U. (2015) Wave motion in a heavy compressible fluid: revisited. *Eur. J. Mech. B/Fluids*, **49**, 50–57.

Kadri, U. & Stiassnie, M. (2012) Acoustic-gravity waves interacting with the shelf break. *J. Geophys. Res.*, **117**(C3), C03035.

Linton, C. M. & McIver, P. (2001) *Handbook of Mathematical Techniques for Wave–structure Interactions,* 1st edn. CRC Press, 316 pp.

Longuet-Higgins, M. S. (1950) A theory of the origin of microseisms. *Philos. Trans. R. Soc. A Math. Phys. Eng. Sci.*, **243**, 1–35.

Mei, C. C. & Black, J. L. (1969) Scattering of surface waves by rectangular obstacles in waters of finite depth. *J. Fluid Mech.*, **38**, 499–511.

Mei, C. C., Stiassnie, M. & Yue, D. K.-P. (2005) *Theory and Applications of Ocean Surface Waves,* Expanded edn. World Scientific Publishing Co, 1136 pp.

Miller, P. D. (2006) *Applied Asymptotic Analysis*. Providence, RI: American Mathematical Society.

Mohapatra, S. C., Karmakar, D. & Sahoo, T. (2011) On capillary gravity-wave motion in two-layer fluids. *J. Eng. Math.*, **71**, 253–277.

Nosov, M. (1999) Tsunami generation in compressible ocean. *Phys. Chem. Earth, Part B Hydrol. Ocean. Atmos.*, **24**, 437–441.

Nosov, M. A., Kolesov, S. V., Denisova, A. V., Alekseev, A. B. & Levin, B. V. (2007) On the near-bottom pressure variations in the region of the 2003 Tokachi-Oki tsunami source. *Oceanology*, **47**, 26–32.

Rhodes-Robinson, P. F. (1971) On the forced surface waves due to a vertical wave-maker in the presence of surface tension. *Math. Proc. Cambridge Philos. Soc.*, **70**, 323–337.

Stiassnie, M. (2009) Tsunamis and acoustic-gravity waves from underwater earthquakes. *J. Eng. Math.*, **67**, 23–32.

Stoker, J. J. (1992) *Water Waves*, vol. 36. Hoboken, NJ, USA: John Wiley and Sons, Inc.

Ursell, F. (1947) The effect of a fixed vertical barrier on surface waves in deep water. *Math. Proc. Cambridge Philos. Soc.*, **43**, 374–382.

Ursell, F. (1948) On the waves due to the rolling of a ship. *Q. J. Mech. Appl. Math.*, **1**, 246–252.

Vargaftik, N., Volkov, B. & Voljak, L. (1983) International tables of the surface tension of water. *J. Phys. Chem. Ref. Data*, **12**, 817–820.

Yamamoto, T. (1982) Gravity waves and acoustic waves generated by submarine earthquakes. *Int. J. Soil Dyn. Earthq. Eng.*, **1**, 75–82.

Appendix A. Derivation of the new solution

We describe how the solution to the infinite depth wave-maker problem with compressibility is derived, using essentially the same techniques that lead to the analogous, incompressible solution. Note that separation of variables in the problem (2.2), (2.3) leads to a singular Sturm–Liouville problem of limit-point type. Defining $\psi(k,z) = k\cos(kz) + K\sin(kz)$, the analogue of Parseval's equality yields

$$\delta(z-\zeta) = 2K\,\mathrm{e}^{Kz}\,\mathrm{e}^{K\zeta} + \frac{2}{\pi}\int_0^\infty \frac{1}{k^2+K^2}\psi(k,z)\psi(k,\zeta)\,\mathrm{d}k, \tag{A.1}$$

exactly as found in the derivation of Havelock's wave-maker theorem for pure-gravity waves (see Friedman, 1990). This yields the general expansion for the potential

$$\phi(x,z) = 2K\,\mathrm{e}^{Kz} A_0 + \frac{2}{\pi}\int_0^\infty A(k)(k\cos(kz) + K\sin(kz))\,\mathrm{d}k, \tag{A.2}$$

where

$$A_0(x) = \int_{-\infty}^0 \phi(x,\zeta)\,\mathrm{e}^{K\zeta}\,\mathrm{d}\zeta, \tag{A.3}$$

$$A(x;k) = \int_{-\infty}^0 \frac{\phi(x,\zeta)}{K^2+k^2}(k\cos(k\zeta) + K\sin(k\zeta))\,\mathrm{d}\zeta. \tag{A.4}$$

Asking that ϕ above fulfill the Helmholtz equation (2.2) implies

$$A_{0xx} + \left(K^2 + \frac{\omega^2}{c_\mathrm{s}^2}\right)A_0 = 0, \tag{A.5}$$

$$A_{xx} - \left(k^2 - \frac{\omega^2}{c_\mathrm{s}^2}\right)A = 0. \tag{A.6}$$

The first of these has a general solution

$$A_0(x) = a_0\,\mathrm{e}^{\mathrm{i}x\left(K^2+\omega^2/c_\mathrm{s}^2\right)^{1/2}} + b_0\,\mathrm{e}^{-\mathrm{i}x\left(K^2+\omega^2/c_\mathrm{s}^2\right)^{1/2}} \tag{A.7}$$

which clearly represents progressive waves moving rightward (a_0) and leftward (b_0). The general solution to the second equation is

$$A(x;k) = a(k)\,\mathrm{e}^{x(k^2-\omega^2/c_\mathrm{s}^2)^{1/2}} + b(k)\,\mathrm{e}^{-x\left(k^2-\omega^2/c_\mathrm{s}^2\right)^{1/2}}. \tag{A.8}$$

The Sommerfeld radiation condition then yields (3.3). The Neumann boundary conditions for the wave-maker $\phi_x(0,z) = U(z)$ yields

$$A_{0x}(0) = \int_{-\infty}^0 \phi_x(0,\zeta)\,\mathrm{e}^{K\zeta}\,\mathrm{d}\zeta,$$

$$A_x(0;k) = \int_{-\infty}^0 \frac{\phi_x(0,\zeta)}{K^2+k^2}(k\cos k\zeta + K\sin k\zeta)\,\mathrm{d}\zeta,$$

which may be used to specify the coefficients a_0, $a(k)$ and $b(k)$.

Appendix B. Derivation of the asymptotic expression for acoustic-gravity waves

We present the derivation of the asymptotics for the acoustic-gravity modes (4.3). Recall

$$\phi_\mathrm{ag}(x,z) = \frac{2}{\pi}\int_0^{k_\mathrm{s}} a(k)\,\mathrm{e}^{\mathrm{i}x\left(k_\mathrm{s}^2-k^2\right)^{1/2}}(k\cos(kz) + K\sin(kz))\,\mathrm{d}k. \tag{B.1}$$

Decomposing the cos and sin terms into exponentials, we find the above equal to

$$\phi_{\mathrm{ag}} = \frac{1}{\pi}\left(\phi_{\mathrm{ag}}^{-} + \phi_{\mathrm{ag}}^{+}\right) = \frac{1}{\pi}\left(\int_0^{k_{\mathrm{s}}} a(k)(k - iK)\,\mathrm{e}^{\mathrm{i}kz+\mathrm{i}\sqrt{k_{\mathrm{s}}^2-k^2}x}\mathrm{d}k \right. \tag{B.2}$$

$$\left. + \int_0^{k_{\mathrm{s}}} a(k)(k + iK)\,\mathrm{e}^{-\mathrm{i}kz+\mathrm{i}\sqrt{k_{\mathrm{s}}^2-k^2}x}\,\mathrm{d}k\right). \tag{B.3}$$

We use the substitution

$$\sqrt{k_{\mathrm{s}}^2 - k^2} = \mu\cos\theta, \quad k = \mu\sin\theta,$$

write x and z in polar coordinates $x = r\cos\alpha$ and $z = r\sin\alpha$, and consider the domain $\alpha \in (-\pi/2, 0)$. After applying the formulas for products of trigonometric functions

$$\phi_{\mathrm{ag}}^{-} = \int_0^{\pi/2} f^{-}(\theta)\,\mathrm{e}^{\mathrm{i}rH^{-}(\theta)}\,\mathrm{d}\theta, \tag{B.4}$$

$$\phi_{\mathrm{ag}}^{+} = \int_0^{\pi/2} f^{+}(\theta)\,\mathrm{e}^{\mathrm{i}rH^{+}(\theta)}\,\mathrm{d}\theta, \tag{B.5}$$

where we define $f^{\pm}(\theta) = a(\mu\sin\theta)(\mu\sin\theta \pm iK)\mu\cos\theta$, and $H^{\pm}(\theta) = \mu\cos(\theta \pm \alpha)$. We seek to apply the method of stationary phase (see Miller, 2006) to evaluate the behaviour of these integrals for large values of r, and to this end examine the derivatives of $H^{\pm}$:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}H^{\pm}(\theta) = -\mu\sin(\theta \pm \alpha).$$

We see that ϕ_{ag}^{-} has no points of stationary phase, and thus the leading-order contributions will come from the stationary phase points of ϕ_{ag}^{+}. Indeed, for any α, the point $\theta = -\alpha$ is a point of stationary phase, and the second derivative of H^{+} is negative.

The leading terms in the asymptotic expansion for large radius r are thus

$$\begin{aligned}\phi_{\mathrm{ag}} &\sim \frac{1}{\pi}\left(\frac{2\pi}{r|H^{+(2)}(-\alpha)|}\right)^{1/2} a(\mu\sin(-\alpha))(\mu\sin(-\alpha) + iK)\cdot\mu\cos(-\alpha)\,\mathrm{e}^{\mathrm{i}rH^{+}(-\alpha)-\mathrm{i}\pi/4}\\ &= \frac{1}{\pi}\left(\frac{2\pi}{rk_{\mathrm{s}}}\right)^{1/2} a(-k_{\mathrm{s}}\sin(\alpha))(iK - k_{\mathrm{s}}\sin(\alpha))\cdot k_{\mathrm{s}}\cos(\alpha)\,\mathrm{e}^{\mathrm{i}rk_{\mathrm{s}}-\mathrm{i}\pi/4},\end{aligned}$$

leading to (4.4).