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On Zakharov's kernel and the interaction of non-collinear wavetrains in finite water depth

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The non-uniqueness of Zakharov's kernel $T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b)$ for gravity waves in water of finite depth is resolved. This goal is achieved by the physical insight gained from the study of the induced mean flow generated by two interacting wavetrains.

1. Introduction

Zakharov's equation (Zakharov 1968; Zakharov & Kharitonov 1970; Lavrova 1983) is the main existing model to study the temporal, weakly nonlinear, evolution of sea states with a broad band of wavelengths and directions. Zakharov's equation is deterministic; i.e. no stochastic assumptions were made in the course of its derivation. However, it is also a very convenient starting point for the derivation of Hasselmann's stochastic model (Hasselmann 1962; Herterich & Hasselmann 1980).

Zakharov's equation was derived for any constant water depth, but most of its applications, so far, have been for infinitely deep water. This may be due to the fact that its kernel $T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_c, \boldsymbol{k}_d)$, when $\boldsymbol{k}_c = \boldsymbol{k}_a$ and $\boldsymbol{k}_d = \boldsymbol{k}_b$, is non-unique for water of finite depth. This non-uniqueness disappears when the depth tends to infinity. This apparent drawback was already discussed in Stiassnie & Shemer (1984) and in Zakharov (1999) and was studied recently for the special case of $T(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a)$ by Janssen & Onorato (2007).

It is important to note that any calculation of nonlinear interactions requires specific values for kernels such as $T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b)$, so that the current state of affairs may hinder further progress.

The aim of this paper is to resolve the above-mentioned difficulty and to provide specific values for $T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b)$. Zakharov's equation and its application to study the interaction of two wavetrains is presented in § 2. Section 3 is devoted to Zakharov's kernel and its split into regular and singular (i.e. non-unique) parts. Section 4 deals with the induced mean flow for the two-wavetrains problem and uses the mean flow results to resolve the non-uniqueness of the kernel. Finally, some concluding remarks are given in § 5.

2. Zakharov's equation and the interaction of two waves

A generalized complex amplitude spectrum is determined from the Fourier transform of the surface elevation $\hat{\eta}$ and the Fourier transform of the velocity

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potential at the free surface $\hat{\phi}^s$, by

$$\beta(\boldsymbol{k},t) = \left(\frac{g}{2\omega(\boldsymbol{k})}\right)^{1/2}\hat{\eta}(\boldsymbol{k},t) + \mathrm{i}\left(\frac{\omega(\boldsymbol{k})}{2g}\right)^{1/2}\hat{\phi}^s(\boldsymbol{k},t), \tag{2.1}$$

where the Fourier transform is given by

$$\hat{f}(\boldsymbol{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}. \tag{2.2}$$

Here $\cdot$ denotes a scalar product; $\boldsymbol{k} = (k_x, k_y)$ is the wavenumber vector; $\boldsymbol{x} = (x, y)$ are the horizontal space coordinates; and t is time. The function β is assumed to consist of free dominant components B and less-dominating bound components $B', \ldots,$ such that

$$\beta(\boldsymbol{k},t) = (B(\boldsymbol{k},t) + B'(\boldsymbol{k},t) + \cdots)\mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{k})t}. \tag{2.3}$$

The slow temporal evolution of the free dominant components B of a weakly nonlinear wave field is governed by Zakharov's equation

$$\mathrm{i}\frac{\partial B}{\partial t} = \iiint T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)B_1^*B_2B_3\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3, \tag{2.4}$$

where ω is the angular frequency in water of constant depth h, given by the dispersion relation

$$\omega^2 = gk\tanh(kh), \tag{2.5}$$

g being the acceleration due to gravity and $k = |\boldsymbol{k}| = (k_x^2 + k_y^2)^{1/2}$ being the length of the wavenumber vector; i is the imaginary unit; $B_j = B(\boldsymbol{k}_j, t)$ and $\omega_j = \omega(\boldsymbol{k}_j)$. The component $B'(\boldsymbol{k}, t)$ can be found in Mei, Stiassnie & Yue (2005).

Consider the interaction of two weakly nonlinear wavetrains, denoted as a and b, by taking

$$B(\boldsymbol{k},t) = B_a(t)\delta(\boldsymbol{k}-\boldsymbol{k}_a) + B_b(t)\delta(\boldsymbol{k}-\boldsymbol{k}_b). \tag{2.6}$$

Substituting (2.6) into (2.4), gives

$$\mathrm{i}\frac{\mathrm{d}B_a}{\mathrm{d}t} = \left\{T(\boldsymbol{k}_a,\boldsymbol{k}_a,\boldsymbol{k}_a,\boldsymbol{k}_a)|B_a|^2 + [T(\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_b) + T(\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_b,\boldsymbol{k}_a)]\,|B_b|^2\right\}B_a, \tag{2.7a}$$

$$\mathrm{i}\frac{\mathrm{d}B_b}{\mathrm{d}t} = \left\{T(\boldsymbol{k}_b,\boldsymbol{k}_b,\boldsymbol{k}_b,\boldsymbol{k}_b)|B_b|^2 + [T(\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_a) + T(\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_a,\boldsymbol{k}_b)]\,|B_a|^2\right\}B_b. \tag{2.7b}$$

The solution of system (2.7) is given by

$$B_a(t) = A_a\exp\left\{-\mathrm{i}\left[T(\boldsymbol{k}_a,\boldsymbol{k}_a,\boldsymbol{k}_a,\boldsymbol{k}_a)A_a^2 + (T(\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_b) + T(\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_b,\boldsymbol{k}_a))A_b^2\right]t\right\}, \tag{2.8a}$$

$$B_b(t) = A_b\exp\left\{-\mathrm{i}\left[T(\boldsymbol{k}_b,\boldsymbol{k}_b,\boldsymbol{k}_b,\boldsymbol{k}_b)A_b^2 + (T(\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_b,\boldsymbol{k}_a) + T(\boldsymbol{k}_b,\boldsymbol{k}_a,\boldsymbol{k}_a,\boldsymbol{k}_b))A_a^2\right]t\right\}. \tag{2.8b}$$

Substituting (2.6) and (2.8) into (2.3) and taking the inverse of (2.1) yields

$$\eta(\boldsymbol{x},t) = a_a\cos(\boldsymbol{k}_a\cdot\boldsymbol{x} - \Omega_a t) + a_b\cos(\boldsymbol{k}_b\cdot\boldsymbol{x} - \Omega_b t). \tag{2.9}$$

In (2.9) a_a and a_b represent the amplitudes of the two wavetrains and are related to the constants A_a and A_b by

$$A_a = 2\pi \left(\frac{g}{2\omega_a}\right)^{1/2} a_a, \qquad A_b = 2\pi \left(\frac{g}{2\omega_b}\right)^{1/2} a_b. \tag{2.10}$$

The frequencies of the wavetrains are given by

$$\Omega_a = \omega_a + T(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a)A_a^2 + [T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b) + T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_b, \boldsymbol{k}_a)]\, A_b^2, \tag{2.11a}$$

$$\Omega_b = \omega_b + T(\boldsymbol{k}_b, \boldsymbol{k}_b, \boldsymbol{k}_b, \boldsymbol{k}_b)A_b^2 + [T(\boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a) + T(\boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_b)]\, A_a^2. \tag{2.11b}$$

In the following section, we provide the expression for Zakharov's kernel $T_{0,1,2,3} \equiv T(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ and demonstrate the non-uniqueness of $T_{a,a,a,a} \equiv T(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a)$ and of $T_{a,b,a,b} \equiv T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b)$ for water of finite and constant depth h.

3. Zakharov's kernel

From (14.B.13) in Mei *et al.* (2005), one can show that

$$T_{0,1,2,3} = W_{0,1,2,3} - 2(A_{0,1,2,3} + B_{0,1,2,3} + C_{0,1,2,3}), \tag{3.1}$$

where

$$A_{0,1,2,3} = \frac{V^{(-)}_{3,3-1,1} V^{(-)}_{0,2,0-2}}{\omega_{1-3} - \omega_3 + \omega_1} + \frac{V^{(-)}_{2,0,2-0} V^{(-)}_{1,1-3,3}}{\omega_{1-3} - \omega_1 + \omega_3}, \tag{3.2}$$

$$B_{0,1,2,3} = \frac{V^{(-)}_{2,2-1,1} V^{(-)}_{0,3,0-3}}{\omega_{1-2} - \omega_2 + \omega_1} + \frac{V^{(-)}_{3,0,3-0} V^{(-)}_{1,1-2,2}}{\omega_{1-2} - \omega_1 + \omega_2}, \tag{3.3}$$

$$C_{0,1,2,3} = \frac{V^{(-)}_{0+1,0,1} V^{(-)}_{2+3,2,3}}{\omega_{2+3} - \omega_2 - \omega_3} + \frac{V^{(+)}_{-2-3,2,3} V^{(+)}_{0,1-0-1}}{\omega_{3+2} + \omega_2 + \omega_3} \tag{3.4}$$

and

$$V^{(\pm)}_{0,1,2} = \frac{1}{8\pi}\left\{ \left(\frac{g\omega_2}{2\omega_0\omega_1}\right)^{1/2} \left[\boldsymbol{k}_0 \cdot \boldsymbol{k}_1 \pm \left(\frac{\omega_0\omega_1}{g}\right)^2\right] + \left(\frac{g\omega_1}{2\omega_0\omega_2}\right)^{1/2} \right.$$
$$\left. \times \left[\boldsymbol{k}_0 \cdot \boldsymbol{k}_2 \pm \left(\frac{\omega_0\omega_2}{g}\right)^2\right] + \left(\frac{g\omega_0}{2\omega_1\omega_2}\right)^{1/2} \left[\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 + \left(\frac{\omega_1\omega_2}{g}\right)^2\right] \right\}, \tag{3.5}$$

$$W_{0,1,2,3} = \bar{W}_{-0,-1,2,3} + \bar{W}_{2,3,-0,-1} - \bar{W}_{2,-1,-0,3} - \bar{W}_{-0,2,-1,3} - \bar{W}_{-0,3,2,-1} - \bar{W}_{3,-1,2,-0}, \tag{3.6}$$

$$\bar{W}_{0,1,2,3} = \frac{1}{64\pi^2}\left(\frac{\omega_2\omega_3}{\omega_0\omega_1}\right)^{1/2} k_0 k_1 \left\{ 2k_0 \tanh(k_1 h) + 2k_1 \tanh(k_0 h) \right.$$
$$\left. - \frac{1}{g} \tanh(k_0 h) \tanh(k_1 h) \left[\omega_{0+2}^2 + \omega_{0+3}^2 + \omega_{1+2}^2 + \omega_{1+3}^2\right] \right\}. \tag{3.7}$$

The calculation of $T(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is straightforward except for the two special cases $T(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a)$ and $T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b)$. In order to calculate $T_{a,a,a,a}$ and $T_{a,b,a,b}$, one has to start from $T(\boldsymbol{k}_a + \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_2, \boldsymbol{k}_a + \boldsymbol{\kappa}_3)$ and $T(\boldsymbol{k}_a + \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1, \boldsymbol{k}_b + \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_2, \boldsymbol{k}_b + \boldsymbol{\kappa}_3)$ and let $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ and $\boldsymbol{\kappa}_3$ tend to zero. This elaborate small-perturbations approach is necessary, since some of the terms have simultaneously vanishing numerators and denominators. Here and in the rest of the paper, the regular/singular parts of the kernels are denoted by the superscripts $(R)/(S)$, respectively.

For $T_{a,a,a,a}$ the small-perturbations approach leads to

$$T(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a) = T^{(R)}(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a) + T^{(S)}(\boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a, \boldsymbol{k}_a), \tag{3.8a}$$

where

$$T^{(R)}_{a,a,a,a} = W_{a,a,a,a} - 2C_{a,a,a,a} = \frac{k_a^2}{32\pi^2 g \omega_a^6}\left(9\omega_a^8 - 10g^2k_a^2\omega_a^4 + 9g^4k_a^4\right) \tag{3.8b}$$

and

$$\begin{aligned} T^{(S)}_{a,a,a,a} &= -2 \lim_{\kappa_3,\kappa_2,\kappa_1\to 0}\{A(\boldsymbol{k}_a + \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_2, \boldsymbol{k}_a + \boldsymbol{\kappa}_3) \\ &\quad + B(\boldsymbol{k}_a + \boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_1, \boldsymbol{k}_a + \boldsymbol{\kappa}_2, \boldsymbol{k}_a + \boldsymbol{\kappa}_3)\} \\ &= -\frac{g}{32\pi^2} \lim_{\kappa_3,\kappa_2,\kappa_1\to 0} \sum_{j=2}^{3} \frac{4\left[1 + \dfrac{Cg_a}{k_a\omega_a}\left(k_a^2 - \dfrac{\omega_a^4}{g^2}\right)\right][\boldsymbol{k}_a\cdot(\boldsymbol{\kappa}_j - \boldsymbol{\kappa}_1)]^2 + \left(k_a^2 - \dfrac{\omega_a^4}{g^2}\right)^2 \dfrac{\omega_{j-1}^2}{\omega_a^2}}{\omega_{j-1}^2 - [\boldsymbol{Cg}_a\cdot(\boldsymbol{\kappa}_j - \boldsymbol{\kappa}_1)]^2}. \end{aligned} \tag{3.8c}$$

In the above, $\boldsymbol{Cg}$ is the group velocity. Note that (3.8*b*) and (3.8*c*) are identical to (3.9*b*) and (3.9*c*) in Stiassnie & Shemer (1984). To obtain $T_{b,b,b,b}$, one has to replace $\boldsymbol{k}_a$ by $\boldsymbol{k}_b$ in (3.8). For $T_{a,b,a,b}$ the small-perturbations approach leads to

$$T(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b) = T^{(R)}(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b) + T^{(S)}(\boldsymbol{k}_a, \boldsymbol{k}_b, \boldsymbol{k}_a, \boldsymbol{k}_b), \tag{3.9a}$$

where

$$\begin{aligned} T^{(R)}_{a,b,a,b} &= W_{a,b,a,b} - 2(B_{a,b,a,b} + C_{a,b,a,b}) = \frac{g}{32\pi^2\omega_a\omega_b}\Bigg\{-2\frac{\omega_a^2\omega_b^2}{g^2}\left(k_a^2 + k_b^2\right) \\ &+ \frac{1}{\omega_{a-b}^2 - (\omega_a - \omega_b)^2}\Bigg\{\left[\omega_b\left(k_a^2 - \boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) - \omega_a\left(k_b^2 - \boldsymbol{k}_a\cdot\boldsymbol{k}_b\right)\right] \\ &\times\left[-\omega_b\left(k_a^2 - 3\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + \omega_a\left(k_b^2 - 3\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + 2\frac{\omega_a^2\omega_b^2}{g^2}(\omega_a - \omega_b)\right] \\ &- \left[(\boldsymbol{k}_a\cdot\boldsymbol{k}_b)^2 + 2\frac{\omega_a\omega_b^3}{g^2}\left(k_a^2 - 2\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) - 2\frac{\omega_a^2\omega_b^2}{g^2}\left(k_a^2 + k_b^2 - 3\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right)\right. \\ &\left.+ 2\frac{\omega_a^3\omega_b}{g^2}\left(k_b^2 - 2\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + \frac{\omega_a^2\omega_b^2}{g^4}\left(\omega_a^2 - \omega_a\omega_b + \omega_b^2\right)^2\right]\omega_{a-b}^2\Bigg\} \\ &- \frac{1}{\omega_{a+b}^2 - (\omega_a + \omega_b)^2}\Bigg\{\left[\omega_b\left(k_a^2 + \boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + \omega_a\left(k_b^2 + \boldsymbol{k}_a\cdot\boldsymbol{k}_b\right)\right] \\ &\times\left[\omega_b\left(k_a^2 + 3\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + \omega_a\left(k_b^2 + 3\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + 2\frac{\omega_a^2\omega_b^2}{g^2}(\omega_a + \omega_b)\right] \\ &+ \left[(\boldsymbol{k}_a\cdot\boldsymbol{k}_b)^2 - 2\frac{\omega_a\omega_b^3}{g^2}\left(k_a^2 + 2\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) - 2\frac{\omega_a^2\omega_b^2}{g^2}\left(k_a^2 + k_b^2 + 3\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right)\right. \\ &\left.- 2\frac{\omega_a^3\omega_b}{g^2}\left(k_b^2 + 2\boldsymbol{k}_a\cdot\boldsymbol{k}_b\right) + \frac{\omega_a^2\omega_b^2}{g^4}\left(\omega_a^2 + \omega_a\omega_b + \omega_b^2\right)^2\right]\omega_{a+b}^2\Bigg\}\Bigg\} \end{aligned} \tag{3.9b}$$

and

$$T^{(S)}_{a,b,a,b} = -2 \lim_{\kappa_3,\kappa_2,\kappa_1\to 0} A(\boldsymbol{k}_a+\boldsymbol{\kappa}_3+\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_1, \boldsymbol{k}_b+\boldsymbol{\kappa}_1, \boldsymbol{k}_a+\boldsymbol{\kappa}_2, \boldsymbol{k}_b+\boldsymbol{\kappa}_3)$$
$$= -\frac{g}{16\pi^2} \lim_{\kappa_3,\kappa_1\to 0} \frac{1}{\omega_{3-1}^2 - [\boldsymbol{C}\boldsymbol{g}_b\cdot(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1)]^2} \left\{ \left(k_a^2 - \frac{\omega_a^4}{g^2}\right)\left(k_b^2 - \frac{\omega_b^4}{g^2}\right)\frac{\omega_{3-1}^2}{2\omega_a\omega_b} + \left[2+\frac{Cg_b}{k_b\omega_b}\left(k_b^2-\frac{\omega_b^4}{g^2}\right)\right][\boldsymbol{k}_a\cdot(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1)]\,[\boldsymbol{k}_b\cdot(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1)] + \frac{1}{\omega_a}\left(k_a^2-\frac{\omega_a^4}{g^2}\right)[\boldsymbol{C}\boldsymbol{g}_b\cdot(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1)]\,[\boldsymbol{k}_b\cdot(\boldsymbol{\kappa}_3-\boldsymbol{\kappa}_1)] \right\}. \tag{3.9c}$$

For $T^{(S)}_{a,b,b,a}$ one has to replace $\boldsymbol{\kappa}_3$ by $\boldsymbol{\kappa}_2$ in (3.9*c*). Note that $T^{(R)}_{b,a,b,a} = T^{(R)}_{a,b,a,b}$ and that for $T^{(S)}_{b,a,b,a}$, one needs to interchange a and b in (3.9*c*). For deep water, i.e. $h\to\infty$, all $T^{(S)}$ vanish.

Defining

$$\cos\theta_{aj} = \lim_{\kappa_1,\kappa_j\to 0} \frac{\boldsymbol{k}_a\cdot(\boldsymbol{\kappa}_j-\boldsymbol{\kappa}_1)}{k_a|\boldsymbol{\kappa}_j-\boldsymbol{\kappa}_1|}, \qquad \cos\theta_{bj} = \lim_{\kappa_1,\kappa_j\to 0} \frac{\boldsymbol{k}_b\cdot(\boldsymbol{\kappa}_j-\boldsymbol{\kappa}_1)}{k_b|\boldsymbol{\kappa}_j-\boldsymbol{\kappa}_1|} \tag{3.10}$$

and noting that

$$\lim_{\kappa_1,\kappa_j\to 0} \omega_{j-1}^2 = gh|\boldsymbol{\kappa}_j-\boldsymbol{\kappa}_1|^2, \tag{3.11}$$

(3.8*c*) and (3.9*c*) become

$$T^{(S)}_{a,a,a,a} = -\frac{g}{32\pi^2}\sum_{j=2}^{3} \frac{4k_a^2\left[1+\frac{Cg_a}{k_a\omega_a}\left(k_a^2-\frac{\omega_a^4}{g^2}\right)\right]\cos^2\theta_{aj} + \left(k_a^2-\frac{\omega_a^4}{g^2}\right)^2\frac{gh}{\omega_a^2}}{gh - Cg_a^2\cos^2\theta_{aj}} \tag{3.12}$$

and

$$T^{(S)}_{a,b,a,b} = -\frac{g}{16\pi^2(gh - Cg_b^2\cos^2\theta_{b3})}\left\{\left(k_a^2-\frac{\omega_a^4}{g^2}\right)\left(k_b^2-\frac{\omega_b^4}{g^2}\right)\frac{gh}{2\omega_a\omega_b} + k_ak_b\left[2+\frac{Cg_b}{k_b\omega_b}\left(k_b^2-\frac{\omega_b^4}{g^2}\right)\right]\cos\theta_{a3}\cos\theta_{b3} + \frac{Cg_bk_b}{\omega_a}\left(k_a^2-\frac{\omega_a^4}{g^2}\right)\cos^2\theta_{b3}\right\}. \tag{3.13}$$

Note the non-unique nature of the singular kernels, whose values depend on the directions of approach to zero. For strictly two-dimensional motion the choice of θ_{aj} and θ_{bj} is limited to the two values 0 or π. In this case $\cos^2\theta_{aj} = \cos^2\theta_{bj} = 1$ and $\cos\theta_{a3}\cos\theta_{b3} = 1$ if $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$ are collinear and -1 if $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$ point in opposite directions. Thus for two-dimensional motion the non-uniqueness problem is easily resolved. For the more general case of three-dimensional motion however, additional arguments must be employed in order to resolve the non-uniqueness. In the following section, the induced mean flow is introduced and used to choose appropriate values for θ_{aj} and θ_{bj}.

4. The induced mean flow and the main results

Starting from B' given in (14.3.3) of Mei *et al.* (2005), one can calculate the potential of the induced mean flow produced by a narrow spectrum centred around, say, $\boldsymbol{k}_a$:

$$\bar{\phi}_a(\boldsymbol{x}, z, t) = \frac{\mathrm{i}g}{8\pi^2} \iint \frac{\cosh[|\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1|(h+z)]}{\cosh[|\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1|h]} B^*(\boldsymbol{k}_a + \boldsymbol{\kappa}_1) B(\boldsymbol{k}_a + \boldsymbol{\kappa}_2) \times \frac{\frac{1}{\omega_a}[\boldsymbol{C}g_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]\left(k_a^2 - \frac{\omega_a^4}{g^2}\right) + 2[\boldsymbol{k}_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]}{\omega_{2-1}^2 - [\boldsymbol{C}g_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]^2} \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)\cdot(\boldsymbol{x} - \boldsymbol{C}g_a t)} \,\mathrm{d}\boldsymbol{\kappa}_2 \,\mathrm{d}\boldsymbol{\kappa}_1. \tag{4.1}$$

Similarly one can find the mean free surface $\bar{\eta}_a$:

$$\bar{\eta}_a(\boldsymbol{x}, t) = -\frac{1}{8\pi^2} \iint B^*(\boldsymbol{k}_a + \boldsymbol{\kappa}_1) B(\boldsymbol{k}_a + \boldsymbol{\kappa}_2) \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)\cdot(\boldsymbol{x} - \boldsymbol{C}g_a t)} \times \frac{2[\boldsymbol{C}g_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)][\boldsymbol{k}_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)] + \frac{\omega_{2-1}^2}{\omega_a}\left(k_a^2 - \frac{\omega_a^4}{g^2}\right)}{\omega_{2-1}^2 - [\boldsymbol{C}g_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]^2} \,\mathrm{d}\boldsymbol{\kappa}_2 \,\mathrm{d}\boldsymbol{\kappa}_1. \tag{4.2}$$

From (4.1), the induced current at $z = 0$, $\boldsymbol{u}_a$, is given by

$$\boldsymbol{u}_a(\boldsymbol{x}, t) = -\frac{g}{8\pi^2} \iint B^*(\boldsymbol{k}_a + \boldsymbol{\kappa}_1) B(\boldsymbol{k}_a + \boldsymbol{\kappa}_2) \mathrm{e}^{\mathrm{i}(\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)\cdot(\boldsymbol{x} - \boldsymbol{C}g_a t)} \times \frac{\frac{1}{\omega_a}[\boldsymbol{C}g_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]\left(k_a^2 - \frac{\omega_a^4}{g^2}\right) + 2[\boldsymbol{k}_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]}{\omega_{2-1}^2 - [\boldsymbol{C}g_a \cdot (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1)]^2} (\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1) \,\mathrm{d}\boldsymbol{\kappa}_2 \,\mathrm{d}\boldsymbol{\kappa}_1. \tag{4.3}$$

For a single wavetrain one has $B(\boldsymbol{k}_a + \boldsymbol{\kappa}_j) = B_a(t)\delta(\boldsymbol{\kappa}_j)$, which when substituted into (4.2) and (4.3) produces

$$\boldsymbol{u}_a = -\frac{g k_a A_a^2}{8\pi^2} \frac{\left[\frac{Cg_a}{k_a \omega_a}\left(k_a^2 - \frac{\omega_a^4}{g^2}\right) + 2\right]\cos\theta_{a2}}{gh - Cg_a^2 \cos^2\theta_{a2}} \lim_{\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \to 0} \frac{\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1}{|\boldsymbol{\kappa}_2 - \boldsymbol{\kappa}_1|} \tag{4.4}$$

and

$$\bar{\eta}_a = -\frac{A_a^2}{8\pi^2} \frac{2Cg_a k_a \cos^2\theta_{a2} + \frac{gh}{\omega_a}\left(k_a^2 - \frac{\omega_a^4}{g^2}\right)}{gh - Cg_a^2 \cos^2\theta_{a2}}, \tag{4.5}$$

where θ_{a2} is defined in (3.10). Similar expressions exist for $\boldsymbol{u}_b$ and $\bar{\eta}_b$.

From (3.12), (4.4) and (4.5), one can see that

$$T^{(S)}_{a,a,a,a} A_a^2 = \boldsymbol{k}_a \cdot \boldsymbol{u}_a + \frac{g}{2\omega_a}\left(k_a^2 - \frac{\omega_a^4}{g^2}\right)\bar{\eta}_a. \tag{4.6}$$

Similarly from (3.13), (4.4) and (4.5), one gets

$$\left[T^{(S)}_{a,b,a,b} + T^{(S)}_{a,b,b,a}\right] A_b^2 = \boldsymbol{k}_a \cdot \boldsymbol{u}_b + \frac{g}{2\omega_a}\left(k_a^2 - \frac{\omega_a^4}{g^2}\right)\bar{\eta}_b. \tag{4.7}$$

Substituting (4.6) and (4.7) into (2.11*a*) gives

$$\Omega_a = \omega_a + T^{(R)}_{a,a,a,a}A_a^2 + 2T^{(R)}_{a,b,a,b}A_b^2 + \boldsymbol{k}_a\cdot(\boldsymbol{u}_a+\boldsymbol{u}_b) + \frac{g}{2\omega_a}\left(k_a^2-\frac{\omega_a^4}{g^2}\right)(\bar{\eta}_a+\bar{\eta}_b). \quad (4.8)$$

Note that the contributions of the wave $\boldsymbol{k}_b$ and both its induced mean flow $\boldsymbol{u}_b$ and its mean elevation $\bar{\eta}_b$ appear in (4.8) in a somewhat expected and rather symmetric fashion. For a single wavetrain, say a, one would expect the induced flow to be collinear with $\boldsymbol{k}_a$; i.e. all $\boldsymbol{\kappa}_j$ must be parallel to $\boldsymbol{k}_a$. This is equivalent to the choice $\cos\theta_{aj}=1$ in (4.4) and (4.5), which then reduce to

$$\boldsymbol{u}_a = -\frac{g\boldsymbol{k}_a A_a^2}{8\pi^2}\,\frac{\left[\dfrac{Cg_a}{k_a\omega_a}\left(k_a^2-\dfrac{\omega_a^4}{g^2}\right)+2\right]}{gh-Cg_a^2}, \quad (4.9a)$$

$$\bar{\eta}_a = -\frac{A_a^2}{8\pi^2}\,\frac{2Cg_a k_a + \dfrac{gh}{\omega_a}\left(k_a^2-\dfrac{\omega_a^4}{g^2}\right)}{gh-Cg_a^2}. \quad (4.9b)$$

One can show that $\boldsymbol{u}_a$, $\bar{\eta}_a$ in (4.9) are identical to β, b in (16.99) of Whitham (1974). Moreover, for a single wavetrain, (4.8), with the above $\boldsymbol{u}_a$, $\bar{\eta}_a$, becomes identical to (16.103) in Whitham (1974).

Introducing $\cos\theta_{aj}=1$ into (3.12) gives

$$T^{(S)}_{a,a,a,a} = -\frac{g}{16\pi^2}\,\frac{4k_a^2\left[1+\dfrac{Cg_a}{k_a\omega_a}\left(k_a^2-\dfrac{\omega_a^4}{g^2}\right)\right]+\left(k_a^2-\dfrac{\omega_a^4}{g^2}\right)\dfrac{gh}{\omega_a^2}}{gh-Cg_a^2}. \quad (4.10)$$

Similarly, one would expect $\boldsymbol{u}_b$ to be collinear with $\boldsymbol{k}_b$, leading to $\cos\theta_{b3}=1$ in (3.13); $\theta_{b3}=0$ together with (3.10) leads to $\theta_{a3}=\theta_{ab}$, where θ_{ab} is the angle between $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$. With these (3.13) takes the form

$$T^{(S)}_{a,b,a,b} = -\frac{g}{16\pi^2(gh-Cg_b^2)}\left\{\left(k_a^2-\frac{\omega_a^4}{g^2}\right)\left(k_b^2-\frac{\omega_b^4}{g^2}\right)\frac{gh}{2\omega_a\omega_b}\right.$$
$$\left. + k_a k_b\left[2+\frac{Cg_b}{k_b\omega_b}\left(k_b^2-\frac{\omega_b^4}{g^2}\right)\right]\cos\theta_{ab} + \frac{Cg_b k_b}{\omega_a}\left(k_a^2-\frac{\omega_a^4}{g^2}\right)\right\}. \quad (4.11)$$

Equations (4.10) and (4.11) are the main results and goal of this paper; together with $T^{(R)}_{a,a,a,a}$ and $T^{(R)}_{a,b,a,b}$ (see (3.8*b*) and (3.9*b*)) and $T^{(S)}_{a,b,a,b}=T^{(S)}_{a,b,b,a}$, they provide explicit expressions for Zakharov's kernels $T_{a,a,a,a}$ and $T_{a,b,a,b}$ in water of finite depth. Note that the resulting expression for $T_{a,a,a,a}$ is identical to (3.4) in Janssen & Onorato (2007). However, expression (4.11), to the best of our knowledge, is new.

In the case of infinite depth, $kh\to\infty$, the induced mean flow and the mean surface vanish; so do $T^{(S)}_{a,a,a,a}$ and $T^{(S)}_{a,b,a,b}$. In this case it can be shown that the nonlinear dispersion relation (4.8) with the deep-water limits of $T^{(R)}_{a,a,a,a}$ and $T^{(R)}_{a,b,a,b}$ corresponds to the results of Longuet-Higgins & Phillips (1962), except for a misprint which was corrected by Hogan, Gruman & Stiassnie (1988).

Recently Madsen & Fuhrman (2006) also considered the interaction of non-collinear, bichromatic waves in finite depth. In their (42) they provide an expression for the nonlinear dispersion relation which is identical to (4.8) when setting their current equal to the induced mean flow, i.e. $\boldsymbol{U}=\boldsymbol{u}_a+\boldsymbol{u}_b$, and assuming zero mean free surface $\bar{\eta}_a=\bar{\eta}_b=0$ in our (4.8). Thus, our results for the regular parts of the free mode interacts with $\boldsymbol{k}_a$ (and $\boldsymbol{k}_b$) in a triad interaction, which results in replacing $(\boldsymbol{u}_a+\boldsymbol{u}_b)$ and $(\bar\eta_a+\bar\eta_b)$ in (4.8) by $(\boldsymbol{u}_a+\boldsymbol{u}_b+\boldsymbol{U})$ and $(\bar\eta_a+\bar\eta_b+N)$, respectively.

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Appendix. Triad interaction with an infinitely long wave

A constant change in mean surface and a uniform current can be described by the surface elevation $\eta^l=N$ and velocity potential $\phi^l=\boldsymbol{U}\cdot\boldsymbol{x}$, where N and $\boldsymbol{U}$ are constants. In the context of the Zakharov equation this may be represented as a 'wave' with the wavenumber vector $\boldsymbol{k}=0$. By using (2.2), one can find the Fourier transforms of ϕ^l and η^l as

$$\hat\phi^l(\boldsymbol{k})=2\pi\mathrm{i}[U_x\delta'(k_x)\delta(k_y)+U_y\delta(k_x)\delta'(k_y)],\qquad \hat\eta^l(\boldsymbol{k})=2\pi N\delta(\boldsymbol{k}),\tag{A 1}$$

where $\delta'(k)$ is the derivative of the Dirac δ function.

In the derivation of the Zakharov equation for gravity waves it is assumed that there is no resonant triad interaction. Hence, triad interactions are removed from the evolution equation by a transformation and appear only in the expressions for bound waves. However, if one accounts for an infinitely long wave, resonant triad interactions are present. By including the possibility of resonant triad interaction in the Zakharov equation one obtains (see e.g. (2.21) in Krasitskii 1994)

$$\mathrm{i}\frac{\partial b}{\partial t}-\omega(\boldsymbol{k})b=\iint V^{(-)}_{0,1,2}b_1b_2\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2 \\ +2\iint V^{(-)}_{2,1,0}b_1^*b_2\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\mathrm{d}\boldsymbol{k}_2,\tag{A 2}$$

where the standard cubic term has been omitted. Here $b(\boldsymbol{k},t)=B(\boldsymbol{k},t)\mathrm{e}^{-\mathrm{i}\omega t}$, where $B(\boldsymbol{k},t)$ is defined in (2.3). We now make use of (2.1) and write $b(\boldsymbol{k},t)$ as a sum of a wavetrain with wavenumber vector $\boldsymbol{k}_a$ and an infinitely long wave as given in (A 1):

$$b(\boldsymbol{k},t)=B_a(t)\mathrm{e}^{-\mathrm{i}\omega_a t}\delta(\boldsymbol{k}-\boldsymbol{k}_a)+\sqrt{\frac{g}{2\omega}}\hat\eta^l+\mathrm{i}\sqrt{\frac{\omega}{2g}}\hat\phi^l.\tag{A 3}$$

Introducing this into (A 2) gives

$$\mathrm{i}\frac{\mathrm{d}B_a}{\mathrm{d}t}=2B_a\int S^{(1)}(\boldsymbol{k}_1)\hat\eta^l_1+\mathrm{i}S^{(2)}(\boldsymbol{k}_1)\hat\phi^l_1\,\mathrm{d}\boldsymbol{k}_1,\tag{A 4}$$

where

$$S^{(1)}(\boldsymbol{k})=\sqrt{\frac{g}{2\omega(\boldsymbol{k})}}\left[V^{(-)}(\boldsymbol{k}_a+\boldsymbol{k},\boldsymbol{k},\boldsymbol{k}_a)+V^{(-)}(\boldsymbol{k}_a,-\boldsymbol{k},\boldsymbol{k}_a+\boldsymbol{k})\right],\tag{A 5a}$$

$$S^{(2)}(\boldsymbol{k})=\sqrt{\frac{\omega(\boldsymbol{k})}{2g}}\left[V^{(-)}(\boldsymbol{k}_a+\boldsymbol{k},\boldsymbol{k},\boldsymbol{k}_a)-V^{(-)}(\boldsymbol{k}_a,-\boldsymbol{k},\boldsymbol{k}_a+\boldsymbol{k})\right].\tag{A 5b}$$

Substituting (A 1) into (A 4) and integrating give

$$\mathrm{i}\frac{\mathrm{d}B_a}{\mathrm{d}t}=4\pi B_aNS^{(1)}(0)+4\pi B_aU_x\left.\frac{\partial S^{(2)}(\boldsymbol{k})}{\partial k_x}\right|_{\boldsymbol{k}=0}+4\pi B_aU_y\left.\frac{\partial S^{(2)}(\boldsymbol{k})}{\partial k_y}\right|_{\boldsymbol{k}=0}.\tag{A 6}$$

Finally, by using (3.5) and solving (A 6), one obtains a frequency correction having the same structure as (4.8), i.e.

$$B_a(t) = A_a \exp\left[-\mathrm{i}\left(\boldsymbol{k}_a \cdot \boldsymbol{U} + \frac{g}{2\omega_a}\left(k_a^2 - \frac{\omega_a^4}{g^2}\right) N\right)\right]. \tag{A 7}$$

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