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On the interaction of four water-waves

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Abstract

The mathematical and statistical properties of the evolution of a system of four interacting surface gravity waves are investigated in detail. Any deterministic quartet of waves is shown to evolve recurrently, but the ensemble averages taken over many realizations with random initial conditions reach constant asymptotic values. The characteristic time-scale for which such asymptotic values are approached is extremely large when randomness is introduced through the initial phases. The characteristic time-scale becomes of an order comparable to that of the recurrence periods when beside the random initial phases, the initial amplitudes are taken to be Rayleigh-distributed. The ensemble-averaged results in the second case resemble, to a certain extent, those derived from the kinetic equation.

Keywords: Water waves; Kinetic equation; Nonlinear interactions

1. Introduction

The dominance of quartet interactions in the nonlinear evolution of surface gravity waves was first established by Phillips [1]. The quartet interaction serves as the building brick in almost any model dealing with spectral evolution. The present work is based on Zakharov [2] equation for the temporal evolution of the complex amplitude spectrum. Zakharov's equation is deterministic, i.e. no stochastic assumptions were made in the course of its derivation, and it is phase-resolving. Earlier, Hasselmann [3] has derived an equation for the nonlinear evolution of the gravity-wave energy spectrum. In the derivation of Hasselmann's equation certain assumptions about the stochastic properties of the system are necessary.

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These assumptions are often referred to by notions, such as: 'nearly Gaussian process', or/and 'random phase approximation'. Details about application of these concepts to Zakharov's equation in order to derive from it the Hasselmann equation can be found in a recent paper by Janssen [4]. The Hasselmann equation, sometimes called the kinetic equation, does not include information about the modal phases, and in this sense it is a phase-averaged equation.

Although the derivation of the kinetic equation does not explicitly require a large number of modes, it may be taken as implied through the assumption of 'near Gaussianity' which is expected to be maintained throughout the evolution, due to the central limit theorem.

The present work focuses on the evolution characteristics of an integrable system of four waves. The advantage in considering a four-wave system is that it allows a closed analytic solution. Such a solution can be used to examine the evolution parameters of a deterministic four-wave system for numerous initial conditions and for any instant, without the need for numerical integration of the governing equations. Moreover, by fixing some initial parameters and varying others, the statistics of the solution can be effectively studied, thus allowing direct estimates of the asymptotic behavior at long times.

A system of four ODEs is obtained from the Zakharov equation in Section 2 (similar equations have been derived by the multiple scale method by [5]). Applying the technique of Bretherton [6], see also Shemer and Stiassnie [7], the detailed solution and its qualitative properties are given in Sections 3–5, followed by a numerical example in Section 6. Stochastic aspects of the four-wave system evolution are raised in Sections 7 and 8. In Section 7 the initial phases are assumed to be random and uniformly distributed, while in Section 8, Rayleigh distributed initial amplitudes are added. Comparison with solutions of Janssen's [4] version of the kinetic equation, for four modes, are given in Section 9. Discussion of the results is presented in Section 10.

2. Formulation

The nonlinear evolution of gravity-wave fields is dominated by the interactions of quartets [1]. The latter is reflected in the structure of Zakharov's equation.

$$i\frac{\partial B_0}{\partial t} = \iiint_{-\infty}^{\infty} V_{0,1,2,3}\, B_1^* B_2 B_3\, \delta_{0+1-2-3}\, \mathrm{e}^{i\Delta_{0,1,2,3}t} \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3 \tag{2.1}$$

which has the quartets 'built-in', through the Delta-function and the frequency detuning

$$\Delta_{0,1,2,3} = \omega_0 + \omega_1 - \omega_2 - \omega_3 \tag{2.2}$$

The interaction coefficients $V_{0,1,2,3} = V(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ are given in Krasitskii [8]. The free-surface elevation is related to the generalized amplitude spectrum $B(\boldsymbol{k},t)$ by

$$\eta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{k}{2\omega}\right)^{1/2} \{B\, \mathrm{e}^{i(kx-\omega t)} + cc\}\mathrm{d}\boldsymbol{k}, \tag{2.3}$$

where the frequency ω is given by the deep-water linear dispersion relation

$$\omega^2 = g|\boldsymbol{k}| \tag{2.4}$$

In this paper we study the evolution of wave-fields, which consist of one quartet of free-waves only, so that

$$B(\boldsymbol{k}, t) = B_a(t)\delta(\boldsymbol{k} - \boldsymbol{k}_a) + B_b(t)\delta(\boldsymbol{k} - \boldsymbol{k}_b) + \boldsymbol{B}_c(t)\delta(\boldsymbol{k} - \boldsymbol{k}_c) + B_d(t)\delta(\boldsymbol{k} - \boldsymbol{k}_d) \tag{2.5}$$

where

$$\boldsymbol{k}_a + \boldsymbol{k}_b - \boldsymbol{k}_c - \boldsymbol{k}_d = 0 \tag{2.6}$$

Substituting (2.5) into (2.1) under the constraint (2.6) gives a system of four first-order nonlinear ordinary differential equations:

$$i\frac{\mathrm{d}B_a}{\mathrm{d}t} = (\Omega_a - \omega_a)B_a + 2V_{abcd}\,\mathrm{e}^{i\Delta_{a,b,c,d}t}B_b^*B_cB_d \tag{2.7a}$$

$$i\frac{\mathrm{d}B_b}{\mathrm{d}t} = (\Omega_b - \omega_b)B_b + 2V_{abcd}\,\mathrm{e}^{i\Delta_{a,b,c,d}t}B_a^*B_cB_d \tag{2.7b}$$

$$i\frac{\mathrm{d}B_c}{\mathrm{d}t} = (\Omega_c - \omega_c)B_c + 2V_{abcd}\,\mathrm{e}^{-i\Delta_{a,b,c,d}t}B_d^*B_aB_b \tag{2.7c}$$

$$i\frac{\mathrm{d}B_\mathrm{d}}{\mathrm{d}t} = (\Omega_d - \omega_d)B_d + 2V_{abcd}\,\mathrm{e}^{-i\Delta_{a,b,c,d}t}B_c^*B_aB_b \tag{2.7d}$$

where the so-called ‘Stokes-corrected’ frequencies are:

$$\Omega_a = \omega_a + V_{aaaa}|B_a|^2 + 2V_{abab}|B_b|^2 + 2V_{acac}|B_c|^2 + 2V_{adad}|B_d|^2 \tag{2.8a}$$

$$\Omega_b = \omega_b + 2V_{baba}|B_a|^2 + V_{bbbb}|B_b|^2 + 2V_{bcbc}|B_c|^2 + 2V_{bdbd}|B_d|^2 \tag{2.8b}$$

$$\Omega_c = \omega_c + 2V_{caca}|B_a|^2 + 2V_{cbcb}|B_c|^2 + V_{cccc}|B_c|^2 + 2V_{cdcd}|B_d|^2 \tag{2.8c}$$

$$\Omega_d = \omega_d + 2V_{dada}|B_a|^2 + 2V_{dbdb}|B_d|^2 + 2V_{dcdc}|B_c|^2 + V_{dddd}|B_d|^2 \tag{2.8d}$$

3. Transition to a single unknown

Multiplying (2.7j) by B_j^*, $j = a, b, c$, and d, and subtracting from the result its complex conjugate yields

$$\frac{\mathrm{d}}{\mathrm{d}t}|B_a|^2 = \frac{\mathrm{d}}{\mathrm{d}t}|B_b|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}|B_c|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}|B_d|^2 = 4V_{abcd}Im\{B_a^*B_b^*B_cB_d\,\mathrm{e}^{i\Delta_{a,b,c,d}t}\} \tag{3.1}$$

An auxiliary real function $Z(t)$ is defined by

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = Im\{B_a^*B_b^*B_cB_d\,\mathrm{e}^{i\Delta_{a,b,c,d}t}\}, \quad Z(0) = 0 \tag{3.2}$$

Substituting (3.2) into (3.1) and integrating gives

$$|B_a|^2 - |\beta_a|^2 = |B_b|^2 - |\beta_b|^2 = -|B_c|^2 + |\beta_c|^2 = -|B_d|^2 + |\beta_d|^2 = 4V_{abcd}Z \tag{3.3}$$

where $\beta_j = B_j(0)$.

From (2.7) and (3.2) one can show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\{B_a^* B_b^* B_c B_d\, \mathrm{e}^{i\Delta_{a,b,c,d}t}\} = -\Omega \frac{\mathrm{d}Z}{\mathrm{d}t} \tag{3.4}$$

where

$$\Omega = \Omega_a + \Omega_b - \Omega_c - \Omega_d \equiv \Omega_0 + \Omega_1 Z \tag{3.5}$$

and Ω_0, Ω_1 are given by

$$\begin{aligned}\Omega_0 = {} & \Delta_{a,b,c,d} + (V_{aaaa} + 2V_{abab} - 2V_{acac} - 2V_{adad})|\beta_a|^2 + (2V_{baba} + V_{bbbb} - 2V_{bcbc} - 2V_{bdbd})|\beta_b|^2 \\ & + (2V_{caca} + 2V_{cbcb} - V_{cccc} - 2V_{cdcd})|\beta_c|^2 + (2V_{dada} + 2V_{dbdb} - 2V_{dcdc} - V_{dddd})|\beta_d|^2 \end{aligned} \tag{3.6a}$$

$$\begin{aligned}\Omega_1 = {} & 4V_{abcd}\{V_{aaaa} + V_{bbbb} + V_{cccc} + V_{dddd} + 4V_{abab} - 4V_{acac} - 4V_{adad} - 4V_{bcbc} - 4V_{bdbd} \\ & + 4V_{cdcd}\}\end{aligned} \tag{3.6b}$$

Integrating (3.4) from $t = 0$ gives

$$Re\{B_a^* B_b^* B_c B_d\, \mathrm{e}^{i\Delta_{a,b,c,d}t}\} = Re\{\beta_a^* \beta_b^* \beta_c \beta_d\} - \int_0^Z \Omega \, \mathrm{d}Z \tag{3.7}$$

From (3.2) and (3.7) we get

$$\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2 = |B_a|^2|B_b|^2|B_c|^2|B_d|^2 - \left[Re\{\beta_a^* \beta_b^* \beta_c \beta_d\} - \Omega_0 Z - \frac{\Omega_1}{2}Z^2\right]^2 \tag{3.8}$$

which is rewritten as

$$\left(\frac{\mathrm{d}Z}{\mathrm{d}t}\right)^2 = P_4(Z) = \sum_{\ell=0}^{4} a_\ell Z^{4-\ell} \tag{3.9}$$

where the coefficients of the forth-order polynomial $P_4(Z)$ are

$$a_0 = -0.25\Omega_1^2 + 256V_{abcd}^4 \tag{3.10a}$$

$$a_1 = -\Omega_0\Omega_1 + 64V_{abcd}^3(|\beta_a|^2 + |\beta_b|^2 - |\beta_c|^2 - |\beta_d|^2) \tag{3.10b}$$

$$\begin{aligned} a_2 = {} & -\Omega_0^2 + \Omega_1|\beta_a\beta_b\beta_c\beta_d|\cos(\arg\beta_a + \arg\beta_b - \arg\beta_c - \arg\beta_d) \\ & + 16V_{abcd}^2(|\beta_a\beta_b|^2 - |\beta_a\beta_c|^2 - |\beta_a\beta_d|^2 - |\beta_b\beta_c|^2 - |\beta_b\beta_d|^2 + |\beta_c\beta_d|^2)\end{aligned} \tag{3.10c}$$

$$\begin{aligned} a_3 = {} & 2\Omega_0|\beta_a\beta_b\beta_c\beta_d|\cos(\arg\beta_a + \arg\beta_b - \arg\beta_c - \arg\beta_d) \\ & - 4V_{abcd}(|\beta_a\beta_b\beta_c|^2 + |\beta_a\beta_b\beta_d|^2 - |\beta_a\beta_c\beta_d|^2 - |\beta_b\beta_c\beta_d|^2)\end{aligned} \tag{3.10d}$$

$$a_4 = |\beta_a\beta_b\beta_c\beta_d|^2\sin^2(\arg\beta_a + \arg\beta_b - \arg\beta_c - \arg\beta_d) \tag{3.10e}$$

4. Periodicity of amplitudes

The formal solution of (3.9) is

$$t = \int_0^Z \frac{\mathrm{d}Z}{\sqrt{P_4(Z)}} \tag{4.1}$$

The actual details depend on the number and values of the real roots of the polynomial; and in principle, various scenarios may exist. However, for the examples that we study herein, we found that $a_0 > 0$ and 4 real roots $Z_4 > Z_3 > 0 > Z_2 > Z_1$. For this case, we apply eq. (255.00) in Byrd and Friedman [9] and obtain from (4.1)

$$a_0^{1/2} t = \gamma \left\{ sn^{-1}\left(\sqrt{\frac{(Z_4 - Z_2)(Z_3 - Z)}{(Z_3 - Z_2)(Z_4 - Z)}}, \kappa \right) + sn^{-1}(\delta, \kappa) \right\} \tag{4.2}$$

where sn is the Jacobian elliptic function with modulus

$$\kappa = \sqrt{\frac{(Z_3 - Z_2)(Z_4 - Z_1)}{(Z_4 - Z_2)(Z_3 - Z_1)}} \tag{4.3}$$

and

$$\gamma = \frac{2}{\sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}} \tag{4.4a}$$

$$\delta = \sqrt{\frac{Z_3(Z_4 - Z_2)}{Z_4(Z_3 - Z_2)}} \tag{4.4b}$$

Inverting (4.2), we find

$$Z = \frac{Z_4(Z_3 - Z_2)sn^2 u - Z_3(Z_4 - Z_2)}{(Z_3 - Z_2)sn^2 u - (Z_4 - Z_2)}; \quad u = sn^{-1}(\delta, \kappa) - a_0^{1/2}\frac{t}{\gamma} \tag{4.5}$$

Utilizing Eqs. (123.01) and (131.01) in Byrd and Friedman [9], we obtain

$$sn(u, \kappa) = \frac{\delta cn(a_0^{1/2} t/\gamma)\, \mathrm{d}n(a_0^{1/2} t/\gamma) - s[(1-\delta^2)(1-\kappa^2\delta^2)]^{1/2} sn(a_0^{1/2} t/\gamma)}{1 - (\kappa\delta)^2 sn^2(a_0^{1/2} t/\gamma)} \tag{4.6}$$

where

$$s = sgn\,(\sin\theta) \tag{4.7a}$$

and

$$\theta = \arg\beta_a + \arg\beta_b - \arg\beta_c - \arg\beta_d \tag{4.7b}$$

Note, that the amplitudes $|B_j|$, $j = a, b, c, d$ in (3.3) depend on Z and thus are periodic, with period

$$T = 2\int_{Z_2}^{Z_3} \frac{\mathrm{d}Z}{\sqrt{P_4}} = \frac{2\gamma}{a_0^{1/2}} sn^{-1}(1,\kappa) = \frac{2\gamma}{a_0^{1/2}} K(\kappa) \tag{4.8}$$

where K is the complete elliptic integral of first kind.

5. Evolution of phases

From (2.7) and (3.3) one can show that

$$\arg B_a = \arg \beta_a - 2V_{abcd}\int_0^t \frac{\beta_0 - \Omega_0 Z - (\Omega_1/2)Z^2}{|\beta_a|^2 + 4V_{abcd}Z}\mathrm{d}t - \int_0^t (\Omega_a - \omega_a)\mathrm{d}t \tag{5.1a}$$

$$\arg B_b = \arg \beta_b - 2V_{abcd}\int_0^t \frac{\beta_0 - \Omega_0 Z - (\Omega_1/2)Z^2}{|\beta_b|^2 + 4V_{abcd}Z}\mathrm{d}t - \int_0^t (\Omega_b - \omega_b)\mathrm{d}t \tag{5.1b}$$

$$\arg B_b = \arg \beta_c - 2V_{abcd}\int_0^t \frac{\beta_0 - \Omega_0 Z - (\Omega_1/2)Z^2}{|\beta_c|^2 - 4V_{abcd}Z}\mathrm{d}t - \int_0^t (\Omega_c - \omega_c)\mathrm{d}t \tag{5.1c}$$

$$\arg B_d = \arg \beta_d - 2V_{abcd}\int_0^t \frac{\beta_0 - \Omega_0 Z - (\Omega_1/2)Z^2}{|\beta_d|^2 - 4V_{abcd}Z}\mathrm{d}t - \int_0^t (\Omega_d - \omega_d)\mathrm{d}t \tag{5.1d}$$

where

$$\beta_0 = |\beta_a\beta_b\beta_c\beta_d| \cos\theta \tag{5.2}$$

From (3.7) we obtain

$$\begin{aligned}\Theta &= \arg B_a + \arg B_b - \arg B_c - \arg B_d \\ &= \Delta_{a,b,c,d}t + \cos^{-1}\frac{\beta_o - \Omega_o Z - (\Omega_1/2)Z^2}{[(4T_{abcd}Z+|\beta_a|^2)(4T_{abcd}Z + |\beta_b|^2)(-4T_{abcd}Z + |\beta_c|^2)(-4T_{abcd}Z + |\beta_d|^2)]^{1/2}}\end{aligned} \tag{5.3}$$

For the free-surface elevation η to be periodic, the following condition must hold for each mode $j = a, b, c, d$

$$\arg B_j(T) - \omega_j T = \arg \beta_j, \tag{5.4}$$

for T given by (4.8).

Numerical results demonstrate that (5.4) does not hold. Thus, the recurrent four-wave system does not exhibit strict periodicity. However, from (5.3) it can be shown that

$$\Theta(T) - \Delta_{a,b,c,d}T = \theta \tag{5.5}$$

For exact resonance conditions, (5.5) demonstrates that the function $\Theta(t)$ itself is periodic. An interesting consequence can be drawn from (5.5) for wave systems consisting of three different waves. This is the case treated by Shemer and Stiassnie [7] who considered degenerated quartets, where $\boldsymbol{k}_b = \boldsymbol{k}_a$, provided that $\boldsymbol{k}_c$ is not collinear with $\boldsymbol{k}_a$ (i.e. the problem is not one-dimensional). In these cases, the constraint (5.5), together with appropriate translation in the horizontal plane, enable to render two snapshots of the free-surface taken at $t = 0$ and $t = T$ identical.

Thus, it seems that to a certain extent, one-dimensional deterministic solutions are “less organized” than their two-dimensional counterparts. Note that this relative “lack of organization” of the one-dimensional deterministic solutions can result in smoother ensemble averages when random initial conditions are added.

6. Example

Generally speaking, the input data consists of eight independent physical quantities: three wave-numbers $\boldsymbol{k}_a$, $\boldsymbol{k}_b$ and $\boldsymbol{k}_c$, four amplitudes $|\beta_a|$, $|\beta_b|\,|\beta_c|$, and $|\beta_d|$, and one phase θ, see (4.7b). In our example, we are starting from a resonating quartet with: $\boldsymbol{k}_a$= (0.9806, −0.1961), $\boldsymbol{k}_b$= (0.9806, 01961), $\boldsymbol{k}_c$ = (1.2903, 0.2747), $\boldsymbol{k}_d = \boldsymbol{k}_a + \boldsymbol{k}_b - \boldsymbol{k}_c$ = (0.6709, −0.2747); and allow variations in the components of $\boldsymbol{k}_c$, $\boldsymbol{k}_d$ in order to move away from exact resonance. The varied $\boldsymbol{k}_c$ and $\boldsymbol{k}_d$ are

$$\boldsymbol{k}_c = (1.2903 - \mu,\ \ 0.2747 + \mu);\ \ \boldsymbol{k}_d = (0.6709 + \mu,\ \ -0.2747 - \mu), \tag{6.1}$$

for $-0.15 < \mu < 0.15$.

Here and in the balance of the paper, we chose the case where $\varepsilon_a = 0.2$, $\varepsilon_b = 0.15$, $\varepsilon_c = 0.08$, $\varepsilon_d = 0.03$. The initial amplitudes are given by

$$|\beta_j| = \frac{\pi\varepsilon_j}{|\boldsymbol{k}_j|}\left(\frac{2g}{\omega_j}\right)^{1/2}, \quad j = a, b, c \text{ and } d \tag{6.2}$$

The initial phase θ is varied in the range $(0, 2\pi)$.

The departure from exact resonance is indicated by a dimensionless scaled detuning parameter

$$\Delta = \Delta_{a,b,c,d}/(\varepsilon_a^2\omega_a) \tag{6.3}$$

The dynamics of the system is captured by two dimensionless quantities: ρ the modulation range, and τ the modulation period. Since in all cases considered the auxiliary function Z varies in the interval $Z_2 \leq Z \leq Z_3$, and in light of (3.3), the modulation range is defined as

$$\rho = \frac{4V_{abcd}(Z_3 - Z_2)}{|\beta_a\beta_b\beta_c\beta_d|^{1/2}}. \tag{6.4}$$

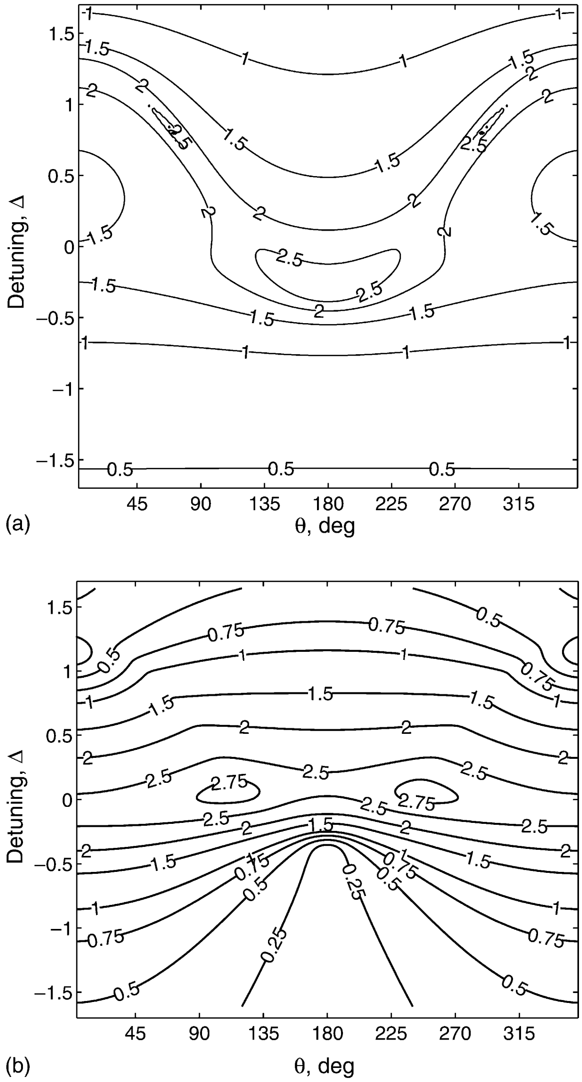

Fig. 1. Isolines of (a) the modulation period τ; (b) the modulation range ρ.

The scaled modulation period is

$$\tau = \frac{\varepsilon_a^2 \omega_a T}{2\pi} \tag{6.5}$$

see Eq. (4.8).

Isolines of τ and ρ are shown in Fig. 1a and b, respectively.

Note that the coordinate $\Delta = 0$ corresponds to the exact resonance conditions. The larger values of $|\Delta|$, at the bottom and top of the figures, correspond to rather weak nonlinear interactions, which are manifested by a substantial decrease in ρ and τ. Both figures indicate a very strong dependence on the initial phase θ. It seems worthwhile to mention that the exact resonance conditions where $\Delta = 0$ are by

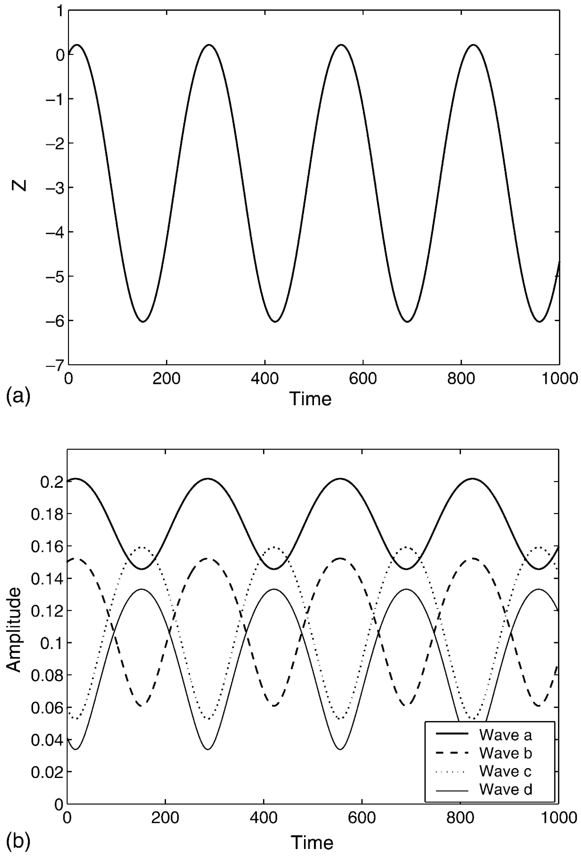

Fig. 2. Evolution of a near-resonant quartet; initial phases: $\arg(\beta_a) = \pi/6$; $\arg(\beta_b) = 0$; $\arg(\beta_c) = -\pi/6$; $\arg(\beta_d) = 0$. (a) auxiliary function Z; (b) wave amplitudes.

no means special, moreover, the extreme values for ρ and τ are obtained for near-resonance, rather than for resonance conditions.

In the following Sections the discussion focuses on the influence of random initial conditions. In order to facilitate the assessment of the results in those Sections, it is instructive to compare them with the plots of deterministic results given in Fig. 2. Fig. 2a and b give the evolution of Z and of the four amplitudes $k_a a_j$, $j = a, b, c, d$, respectively. The dimensionless detuning parameter here is $\varDelta = -0.25$ and the initial phase $\theta = \pi/3$. The time in Fig. 2 and elsewhere in this paper is rendered dimensionless by multiplication of the actual time by ω_a.

7. Random initial phase

Randomness is introduced into the problem through the initial conditions, using the notation

$$B_j(0) = \beta_j = \beta_{jR} + i\beta_{jI} = |\beta_j|\,\mathrm{e}^{i \arg(\beta_j)}, \quad j = a,\ b,\ c,\ d \tag{7.1}$$

Here it is assumed that each of the arg β_j, (j = a, b, c, and d) is uniformly distributed over (0, 2π), but that the amplitudes $|\beta_j|$ are held fixed.

From (3.10) and (5.1), one can see that all $|B_j(t)|$ depend solely on the combination θ, see (4.7b), whereas each arg $B_j(t)$ depends also on its arg (β_j), respectively.

The random initial phases guarantee the property of 'statistical homogeneity', defined as

$$\langle\beta_j\beta_i^*\rangle = |\beta_j|^2\delta_{i,j}, \tag{7.2}$$

where the brackets indicate an ensemble average and $\delta_{i,j}$ is the Kronecker delta. The ensemble average of any quantity Q is defined by

$$\langle Q\rangle = \frac{1}{(2\pi)^4}\iiint\int_{-\pi}^{-\pi} Q\, d(\arg\beta_a)\, d(\arg\beta_b\, d(\arg\beta_c)\, d(\arg\beta_d), \tag{7.3}$$

From the structure of (5.1) and the definition (7.3) one can prove that $B_j(t)$ maintain the property of 'statistical homogeneity', see (7.2), for all time. Moreover all first and third order moments: $\langle B_j\rangle$, $\langle B_jB_iB_k\rangle$, $\langle B_jB_i{}^*B_k\rangle$; are zero.

The periodicity of any single realization of Z, and thus of all $|B_j(t)|$, has been shown in Section 4. In the Appendix we prove that $\langle Z(t)\rangle$, which oscillates initially, settles down at large time to the asymptotic value

$$\langle Z_\infty\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}[Z_1 - \sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}\;\; Z(\beta,\, \kappa)]\,\mathrm{d}\theta, \tag{7.4}$$

where $\beta = \sin^{-1}[(Z_3–Z_1)/(Z_4–Z_1)]$, and $Z(\beta, \kappa)$ is the Zeta function of Jacobi.

The input data here is almost identical to that of Section 6. The exception is that the near-resonant quartet to be discussed here and in sequel includes the following wave vectors: $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$ as before, and $\boldsymbol{k}_c$ = (1.3060, 0.3102), yielding the dimensionless detuning parameter $\Delta = -0.25$, see Eq. (6.3).

The evolution of $\langle Z\rangle$ in time is shown in Fig. 3. The ensemble average was computed over N_p = 2000 initial phases θ. Two time intervals are shown, the first $t \in (0, 5000)$ in Fig. 3a and c, and the second $t \in (10^5, 1.05 \times 10^5)$ in Fig. 3b and d. Note that the instant t = 5000 is reached after about 800 wave-periods. Adopting the notation $t_n = (2\pi/\omega_a)/\varepsilon_a^n$, for the steepness ε under consideration this is well within the t_4 time-scale. Comparing Fig. 3a and b for exact resonance, with Fig. 3c and d for near resonance conditions, one can see that the former are somewhat more erratic, since the evolution of the latter is affected by the detuning frequency. The decay of the oscillations with time is evident for both quartets considered, albeit being extremely slow.

Fig. 4 provides the evolution of the dimensionless averaged amplitudes $\langle A_j\rangle$, j = a, b, c, d, defined as

$$\langle A_j\rangle = \frac{k_a}{\pi}\sqrt{\left(\frac{\omega_j\langle|B_j|^2\rangle}{2g}\right)} \tag{7.5}$$

Naturally, the behavior is very similar to that in Fig. 3, due to the simple relations (3.3).

Very recent computations of Annenkov and Shrira [10] corroborate our results presented in this Section.

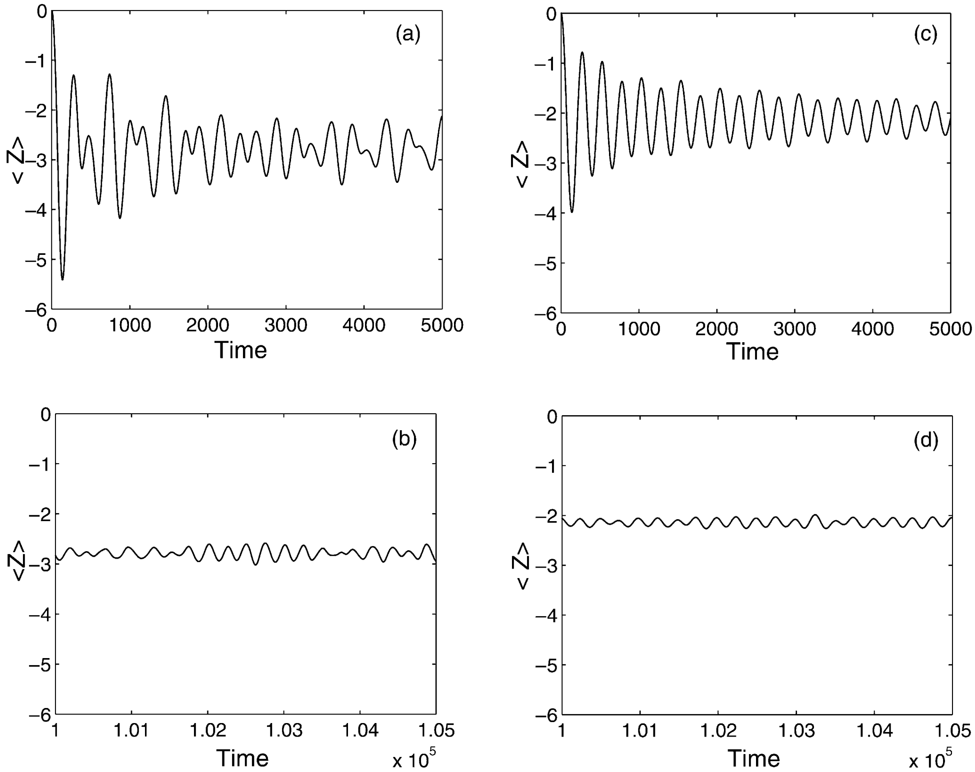

Fig. 3. Evolution of the auxiliary function $\langle Z \rangle$, averaged over 2000 initial phases: (a, b) for resonance conditions; (c, d) for near resonance conditions.

8. Numerical results for random initial amplitudes

As a second exercise we assume that (in addition to random initial phases), the initial amplitudes $|\beta_j|$ are random, and governed by the Rayleigh distribution, so that their pdf is

$$f(|\beta_j|) = \frac{2|\beta_j|}{\langle|\beta_j|^2\rangle} \exp\left(-\frac{|\beta_j|^2}{\langle|\beta_j|^2\rangle}\right), \quad j = a, b, c, d \tag{8.1}$$

The above assumption assures that all β_{jR}, β_{jI}, see Eq. (7.1), have a Gaussian distribution with zero mean and variance $\langle|\beta_j|^2\rangle/2$. The question whether the Gaussianity of $B_j(t) = B_{jR}(t) + iB_{jI}(t)$ is approximately maintained for all time, is an important issue in attempts to develop ‘phase-averaged’ equation, such as the kinetic equation. The values assigned to $\langle|\beta_j|^2\rangle$ in (8.1) are the same as those assigned to $|\beta_j|^2$ in Sections 6 and 7.

In order to render the computations feasible, the number of random initial phases is reduced to $N_p = 50$. The number of realizations of the initial amplitudes for each one of the four modes was set to $N_a = 4$–7. The overall number of realizations in the ensemble is $N_p N_a^4$. The actual initial amplitudes calculated using (8.1) were selected so that each of them has the same probability of $1/N_a$.

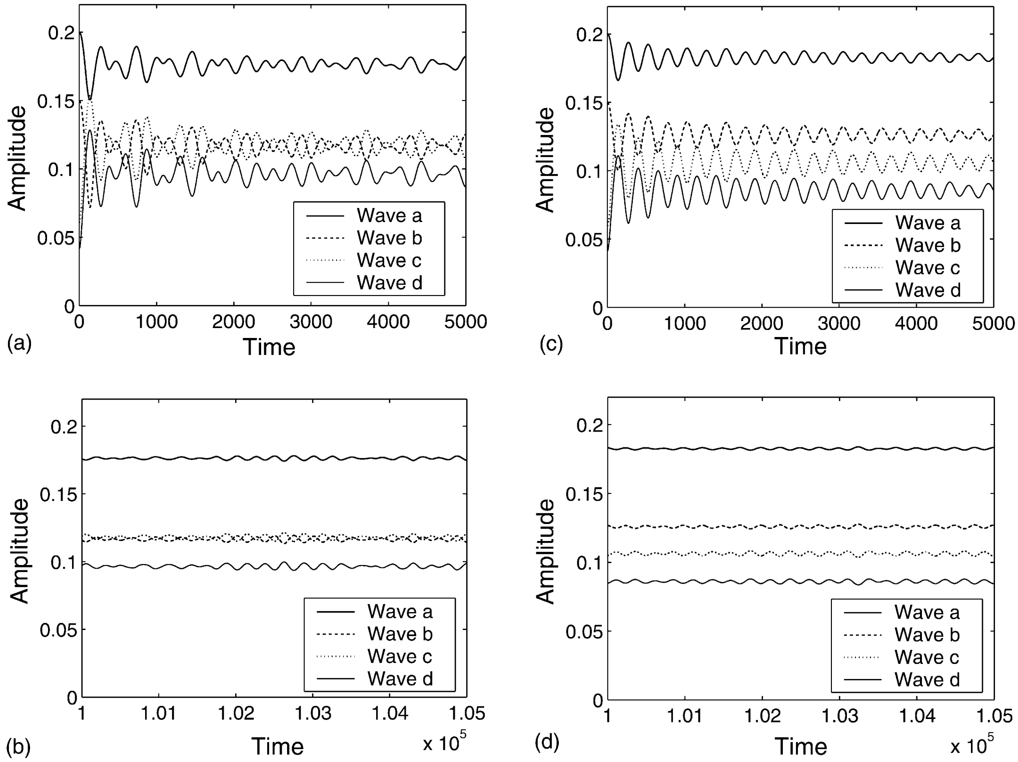

Fig. 4. Evolution of averaged wave-amplitudes (see Eq. (7.5), averaged over 2000 initial phases: (a, b) exact resonance; (c, d) near resonance conditions.

In Fig. 5 the evolution of $\langle Z\rangle$ with random initial phases only is compared to the case where both initial phases and amplitudes are random. The behavior is strikingly different, in the sense that for the latter case nothing interesting occurs for $t > 750 = O(t_3)$.

Computational results for two different numbers of initial amplitude realizations, $N_a = 4$ and 7, are presented in Fig. 6. The agreement between both graphs indicates that the Rayleigh distribution has been satisfactorily approximated.

Fig. 7 demonstrates the similarity between the evolution of the amplitudes for exact resonance condition, to that for the near-resonant case. The similarity in the case of random initial amplitudes should be judged in light of the significant difference observed in Fig. 4, where only random initial phases were considered.

An interesting result of this Section is the fact that the time required for the system to evolve from the initial Gaussian condition to its final stage can be estimated to be of the order between t_2 and t_3, and definitely much shorter than t_4, which is the time-scale implied by Hasselmann's [3] kinetic equation. Similar relatively short evolution timescales have been obtained in some recent papers with a much larger number of modes with initial random phases: Onorato et al. [11] solved the full Euler equation in a box, Janssen [4] studied the one-dimensional nonlinear Schrödinger equation, as well as the Zakharov equation, and Dysthe et al. [12] who performed numerical simulations based on the one-dimensional and two-dimensional modified nonlinear Schrödinger (Dysthe) equation. Similar evolution time-scales were

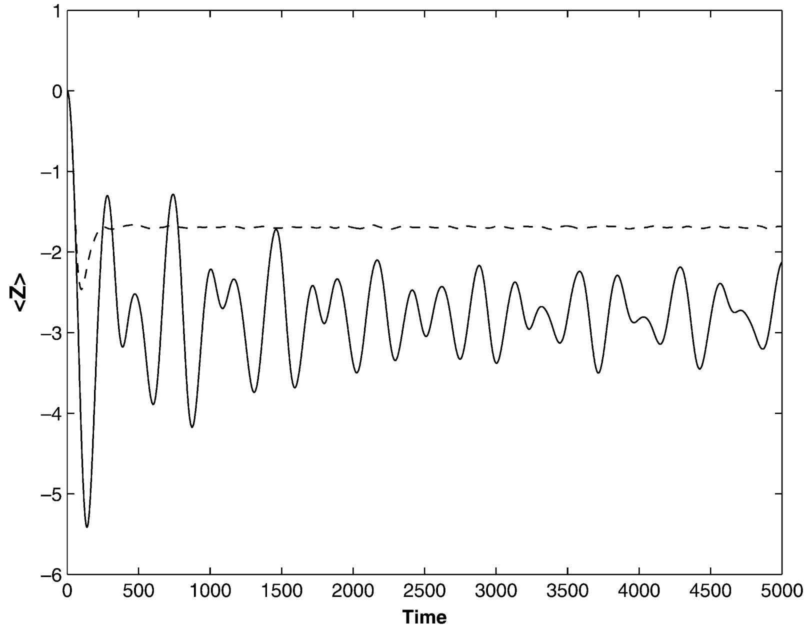

Fig. 5. Evolution of ⟨Z⟩ for a resonant quartet: Solid line—random phases only, N_p = 2000; Dashed line—random amplitudes and phases; N_a = 6, N_p = 50.

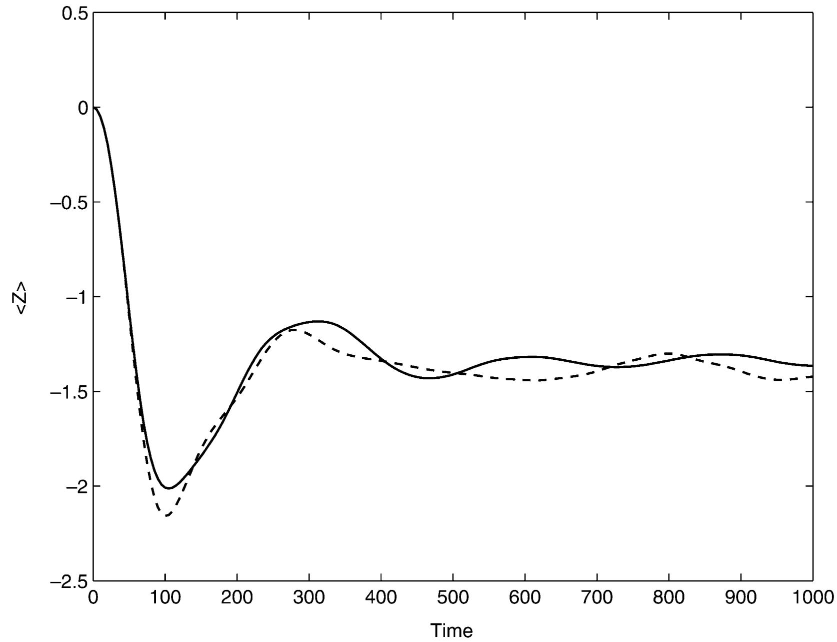

Fig. 6. Evolution of ⟨Z⟩ for near resonance conditions. Number of initial Rayleigh-distributed amplitudes for each mode N_a = 7 (solid line); N_a = 4 (broken line). N_p = 50 in both cases.

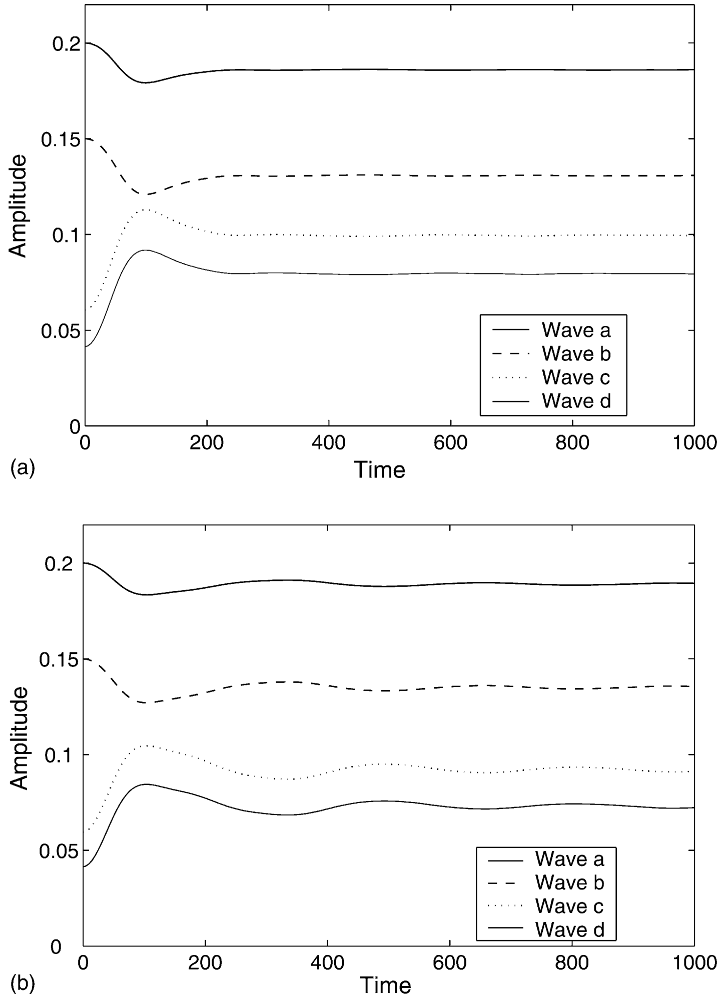

Fig. 7. Evolution of averaged wave amplitudes for $N_p = 50$ and $N_a = 6$ for (a) resonance conditions; (b) near-resonance conditions.

obtained also by Annekov and Shrira [10] who considered a finite number of clusters around wave modes that are in an exact resonance.

9. Comparison with the kinetic equation

Hasselmann [3] was the first to develop a 'phase-averaged' kinetic equation for $C = \langle|B|^2\rangle$. Recently, Janssen [4] has derived the following equation, which contains Hasselmann's result as its long-time asymptotic limit:

$$\frac{\partial C_0}{\partial t} = 4 \iint \int_{-\infty}^{\infty} V_{0,1,2,3}^2 \{C_2C_3(C_0 + C_1) - C_0C_1(C_2 + C_3)\}\, \delta_{0+1-2-3}\, \frac{\sin(\Delta_{0,1,2,3}t)}{\Delta_{0,1,2,3}} \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3 \tag{9.1}$$

Substituting

$$C(\boldsymbol{k},\ t) = C_a\delta(\boldsymbol{k}-\boldsymbol{k}_a) + C_b\delta(\boldsymbol{k}-\boldsymbol{k}_b) + C_c\delta(\boldsymbol{k}-\boldsymbol{k}_c) + C_d\delta(\boldsymbol{k}-\boldsymbol{k}_c) \tag{9.2}$$

into (9.1) yields

$$\frac{\mathrm{d}C_a}{\mathrm{d}t} = \frac{\mathrm{d}C_b}{\mathrm{d}t} = -\frac{\mathrm{d}C_c}{\mathrm{d}t} = -\frac{\mathrm{d}C_d}{\mathrm{d}t} = 8V_{abcd}^2\{C_cC_d(C_a+C_b) - C_aC_b(C_c+C_d)\}\frac{\sin(\Delta_{a,b,c,d}t)}{\Delta_{a,b,c,d}} \tag{9.3}$$

The initial conditions are

$$C_j(0) = c_j, \quad j = a,\ b,\ c,\ d \tag{9.4}$$

The system (9.3) can be reduced to a single equation, say for C_a:

$$\frac{\mathrm{d}C_a}{\mathrm{d}t} = 8V_{abcd}^2\{4C_a^3 + 3(d_b-d_c-d_d)C_a^2 + 2(d_cd_d - d_bd_c - d_bd_d)C_a^2 + d_bd_cd_d\}\frac{\sin(\Delta_{a,b,c,d}t)}{\Delta_{a,b,c,d}} \tag{9.5}$$

where

$$d_b = c_b - c_a; \quad d_c = c_c + c_a; \quad d_d = c_d + c_a; \tag{9.6}$$

Denoting by c_1, c_2, c_3 the roots of the third order polynomial in the curly brackets of (9.5), one can integrate (9.5) to obtain

$$\left[\frac{C_a-c_1}{c_a-c_1}\right] \times \left[\frac{C_a-c_2}{c_a-c_2}\right]^{b_2/b_1} \times \left[\frac{C_a-c_3}{c_a-c_3}\right]^{b_2/b_1} = \exp\left[\frac{32V_{abcd}^2}{b_1\Delta^2}[1-\cos(\Delta_{a,b,c,d}t)]\right] \tag{9.7}$$

where

$$b_1 = \frac{(c_2-c_3)}{\{{c_1}^2(c_2-c_3) + {c_2}^2(c_3-c_1) + {c_3}^2(c_1-c_2)\}} \tag{9.8a}$$

$$b_2 = \frac{(c_3-c_1)}{\{{c_1}^2(c_2-c_3) + {c_2}^2(c_3-c_1) + {c_3}^2(c_1-c_2)\}} \tag{9.8b}$$

$$b_3 = \frac{(c_1-c_2)}{\{{c_1}^2(c_2-c_3) + {c_2}^2(c_3-c_1) + {c_3}^2(c_1-c_2)\}} \tag{9.8c}$$

For exact resonance conditions, one assumes that $\Delta_{a,b,c,d} \to 0$, and (9.7) reduces to

$$\left[\frac{C_a-c_1}{c_a-c_1}\right] \times \left[\frac{C_a-c_2}{c_a-c_2}\right]^{b_2/b_1} \times \left[\frac{C_a-c_3}{c_a-c_3}\right]^{b_2/b_1} = \exp\left(\frac{16\,V_{abcd}^2\,t^2}{b_1}\right) \tag{9.9}$$

To plot $C_a(t)$ one has to solve (9.7) or (9.9). The solutions of (9.9) presented in Fig. 8a are compared with those of (9.7) shown in Fig. 8b. Fig. 8 also includes the corresponding results from Section 8, in Fig. 8c and d, obtained for N_p = 50 and N_a = 6. Generally speaking, Fig. 8b differs qualitatively from the other three figures, at least for $t > 500$, which is $O(t_3)$. From Fig. 8 it is clear that most of the significant changes occur on the scale t_2. Note that there is no quantitative agreement between the averaged results in Fig. 8c and those of the kinetic equation in Fig. 8a. This discrepancy should be related to the understanding that the kinetic equation is valid only for a larger number of modes.

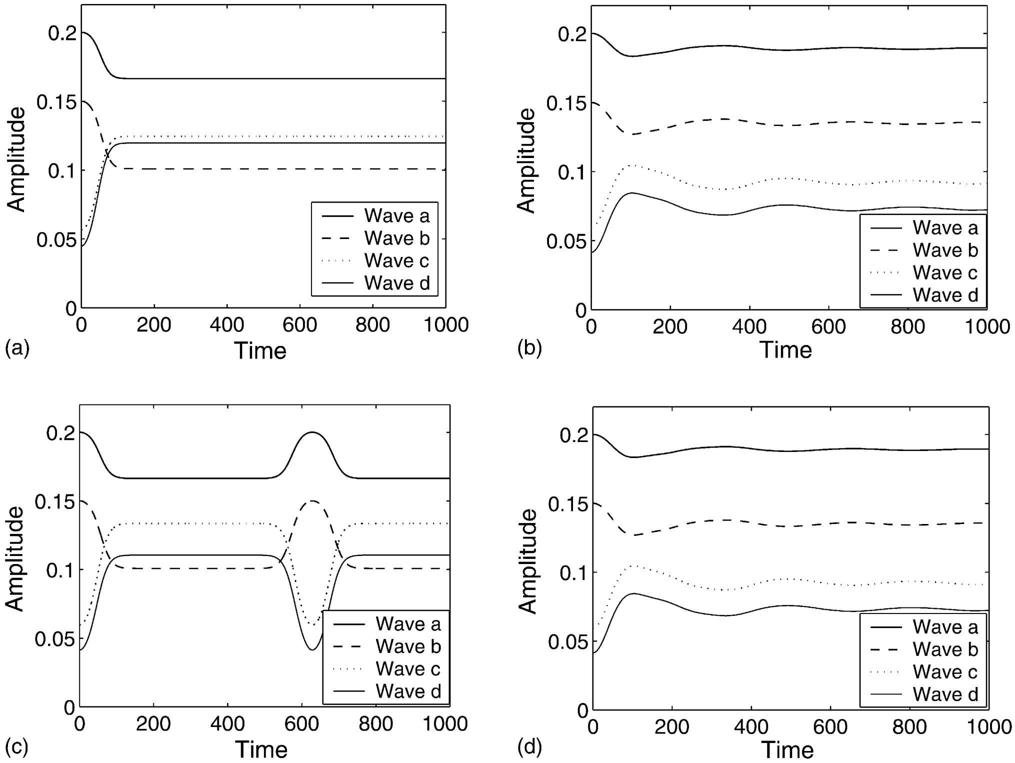

Fig. 8. Evolution of averaged amplitudes: (a) kinetic equation, exact resonance; (b) kinetic equation, near-resonance; (c) and (d) as in (a) and (b), but for an ensemble average with $N_p = 50$ and $N_a = 6$.

Instead of considering implicit solutions (9.7) and (9.9), (9.5) can be integrated numerically using a Runge–Kutta method.

10. Discussion

The main purpose of this section is to shed light on the possible reasons for the significant quantitative differences between the results of the kinetic equation and those of the ensemble average approach, demonstrated in Fig. 8. This requires, however, addressing first some aspects related to the derivation of the kinetic equation.

Taking the ensemble average of (3.1) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle|B_a|^2\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle|B_b|^2\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle|B_c|^2\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle|B_d|^2\rangle = 4V_{abcd}\, Im\{\langle B_a^* B_b^* B_c B_d\rangle\, \mathrm{e}^{i\Delta_{a,b,c,d}t}\} \tag{10.1}$$

In order to find $\langle B_a^* B_b^* B_c B_d\rangle$, the derivative of $B_a^* B_b^* B_c B_d$ is taken, substituting (2.7) yields

$$
\begin{aligned}
-i\frac{\mathrm{d}}{\mathrm{d}t}(B_a^*B_b^*B_cB_d) = {} & \{[V_{aaaa}+2V_{abab}-2V_{acac}-2V_{adad}]|B_a|^2 \\
& +[2V_{abab}+V_{bbbb}-2V_{bcbc}-2V_{bdbd}]|B_b|^2 \\
& +[2V_{acac}+2V_{bcbc}-V_{cccc}\;-\;2V_{cdcd}]|B_c|^2 \\
& +[2V_{adad}+2V_{bdbd}-2V_{cdcd}-V_{dddd}]|B_d|^2\}B_a^*B_b^*B_cB_d \\
& +2V_{abcd}\;\exp(-i\Delta_{a,b,c,d}t)\{|B_a|^2|B_c|^2|B_{\mathrm{d}}|^2+|B_b|^2|B_c|^2|B_d|^2 \\
& -|B_a|^2|B_b|^2|B_c|^2-|B_a|^2|B_b|^2|B_d|^2\} \qquad (10.2)
\end{aligned}
$$

In the course of derivation of the kinetic equation, an ensemble average of both sides of (10.2) is taken, and the following reduced version is adopted:

$$
\begin{aligned}
-i\frac{\mathrm{d}}{\mathrm{d}t}\langle B_a^*B_b^*B_cB_d\rangle = {} & 2V_{abcd}\;\exp(-i\Delta_{a,b,c,d}t)\{\langle|B_a|^2\rangle\langle|B_c|^2\rangle\langle|B_d|^2\rangle+\langle|B_b|^2\rangle\langle|B_c|^2\rangle\langle|B_d|^2\rangle \\
& -\langle|B_a|^2\rangle\langle|B_b|^2\rangle\langle|B_c|^2\rangle-\langle|B_a|^2\rangle\langle|B_b|^2\rangle\langle|B_d|^2\rangle\} \qquad (10.3)
\end{aligned}
$$

See, for example the transition from (18) to (20) in Janssen [4], which makes use of the ‘nearly Gaussian process’ and ‘random phase’ approximations, expressing sixth order moments in terms of products of second order moments. In the case of an integrable system of four-waves, the phases, while initially random, develop a significant coherence in the course of evolution. This is evident from Fig. 9. The cumulative distribution function and the probability density function of Θ (see Eq. (5.3)) at $t = 1000$ are presented in Fig. 9a and b, respectively. The ensemble average was calculated with $N_p = 100$ and $N_a = 7$. More dramatic results were obtained for constant initial amplitudes and $N_p = 50{,}000$, see Fig. 10. Fig. 10c shows the cumulative distribution function at an extreme coherent stage of the evolution process that occurs around $t = 100$.

As a result of the behavior of the phases the RHS of (10.3) is a poor approximation of the ensemble average of the RHS of (10.2) for $t > 20$; this is seen from their plot in Fig. 11.

It is our estimate that comparisons between results obtained by taking ensemble averages over many realizations of a deterministic system will compare more favorably with those from the kinetic equation, when the number of interacting modes in the system is increased. Indeed, Janssen [4] obtained a reasonably good agreement for 51 modes. The fact that we have obtained qualitatively similar behavior for four modes, was rather surprising to us.

Appendix A. on the behavior of $\langle Z(t)\rangle$ at large t (proof of (7.4)).

The study of the deterministic problem yields the solution

$$
Z(\theta,t) = Z_4 - \frac{Z_4-Z_3}{1-(Z_3-Z_2/Z_4-Z_2)sn^2u}; \quad u = sn^{-1}(\delta,\kappa) - a_0^{1/2}\frac{t}{\gamma} \qquad (A.1)
$$

which is periodic in time, with period

$$
T = \frac{2\gamma K(\kappa)}{a_0^{1/2}} \qquad (A.2)
$$

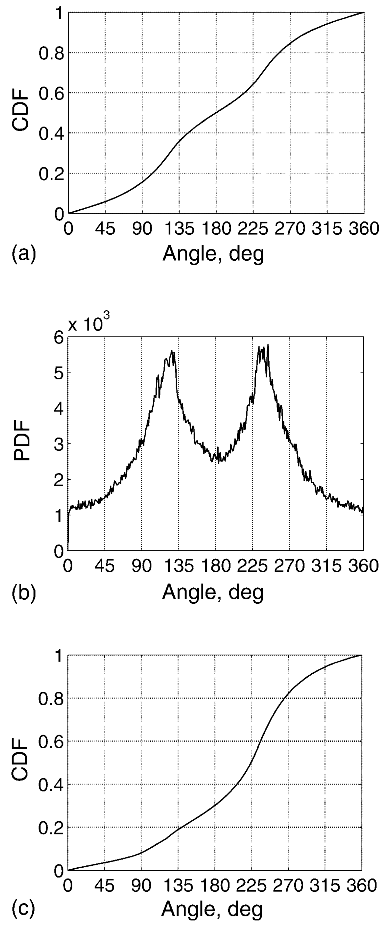

Fig. 9. Resonant quartet, ensemble with $N_p = 100$ and $N_a = 7$: (a) cumulative distribution function of Θ at $t = 1{,}000$; (b) probability density function of Θ at $t = 1000$; (c) as in (a), but for $t = 100$.

and depends on the initial phase difference

$$\theta = \arg\beta_a + \arg\beta_b – \arg\beta_c – \arg\beta_d \tag{A.3}$$

In the stochastic approach we allow random values of θ, with a uniform pdf in $(-\pi, \pi]$. The ensemble average is defined by

$$\langle Z(t)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z(\theta, t)\,\mathrm{d}\theta \tag{A.4}$$

In this Appendix the asymptotic value of $\langle Z\rangle$ is calculated for large t:

$$\langle Z\rangle_\infty = \lim_{t\to\infty}\langle Z\rangle \tag{A.5}$$

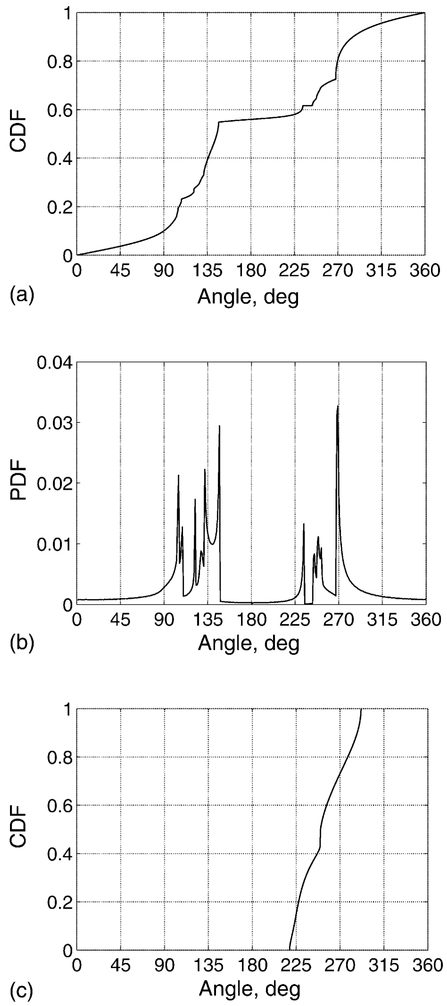

Fig. 10. Resonant quartet, ensemble of 50,000 random initial phases: (a) cumulative distribution function of Θ at $t = 1000$; (b) probability density function of Θ at $t = 1000$; (c) as in (a), but for $t = 100$.

To this end, the periodicity of $Z(\theta, t)$ is utilized. Its Fourier expansion is:

$$Z(\theta, t) = \sum_{n=-\infty}^{\infty} c_n(\theta)\, \mathrm{e}^{2i\pi nt/T(\theta)} \tag{A.6}$$

where

$$c_n(\theta) = \frac{1}{T} \int_0^T Z(\theta, t)\, \mathrm{e}^{-2i\pi nt/T(\theta)}\, \mathrm{d}t \tag{A.7}$$

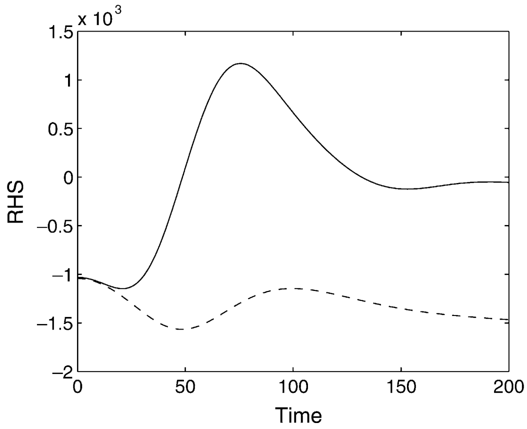

Fig. 11. A comparison of the ensemble average of the RHS of equation (10.2) (solid line), to the RHS of equation (10.3) (dashed line); N_p = 50, N_a = 6.

From (A.4) and (A.6) we obtain

$$\langle Z(t)\rangle = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-\pi}^{\pi} c_n(\theta)\,\mathrm{e}^{2i\pi nt/T(\theta)}\mathrm{d}\theta\right] \tag{A.8}$$

Applying (A.5) to (A.8) yields

$$\langle Z\rangle_\infty = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} C_n, \tag{A.9}$$

where

$$C_n = \lim_{t\to\infty}\int_{-\pi}^{\pi} c_n(\theta)\,\mathrm{e}^{2i\pi nt/T(\theta)}\,\mathrm{d}\theta \tag{A.10}$$

The method of stationary phase guarantees that all C_n, for $n \neq 0$, decay according to

$$C_n \propto \frac{1}{t^{1/2}} \tag{A.11}$$

or faster, see p. 275 in Carrier et al. [13]. Thus, to the leading order, the contribution at large t comes from C_0, i.e.

$$\langle Z\rangle_\infty = \frac{C_0}{2\pi} + O(t^{-1/2}) \tag{A.12}$$

From (A.10) and (A.7)

$$C_0 = \int_{-\pi}^{\pi} c_0(\theta)\,\mathrm{d}\theta \tag{A.13}$$

where

$$c_0 = \frac{1}{T(\theta)} \int_0^T Z(\theta, t)\,\mathrm{d}t \tag{A.14}$$

Substituting (A.1) into (A.14) gives

$$c_0 = Z_4 - \frac{(Z_4 - Z_3)}{T} \int_0^T \frac{\mathrm{d}t}{1 - \alpha^2 sn^2 u} \tag{A.15}$$

where

$$\alpha = \sqrt{\frac{Z_3 - Z_2}{Z_4 - Z_2}} \tag{A.16}$$

Changing the integration variable in (A.15) from t to u, and utilizing (A.2):

$$c_0(\theta) = Z_4 - \frac{Z_4 - Z_3}{2K} \int_{sn^{-1}(\delta,\kappa)-2\,\mathrm{K}}^{sn^{-1}(\delta,\kappa)} \frac{\mathrm{d}u}{1 - \alpha^2 sn^2 u} \tag{A.17}$$

From Byrd and Friedman [9], p. 229, eq. (414.01) we obtain

$$c_0(\theta) = Z_4 - (Z_4 - Z_3)\ \left[1 + \frac{\alpha Z(\beta, \kappa)}{\sqrt{(1-\alpha^2)(\kappa^2 - \alpha^2)}}\right] = Z_3 - \sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}Z(\beta, \kappa) \tag{A.18}$$

where $Z(\beta, \kappa)$ is the Zeta function of Jacobi and

$$\beta = \sin^{-1}\left(\frac{\alpha}{\kappa}\right) = \sin^{-1}\left(\frac{Z_3 - Z_1}{Z_4 - Z_1}\right), \tag{A.19}$$

see Byrd and Friedman [9], p. 33.

Substituting (A.18) into (A.13), and (A.13) into (A.12) gives

$$\langle Z\rangle_\infty = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[Z_3 - \sqrt{(Z_4 - Z_2)(Z_3 - Z_1)}Z(\beta, \kappa)\right] \mathrm{d}\theta \tag{A.20}$$

which is the final result, see Eq. (7.4).

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