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Propagation of Water Waves Over a Source

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ABSTRACT

A simplified problem, in which incoming water waves propagate over a source located beneath the water surface, is treated by analytical techniques. The physical assumptions made are: (i) The flow is potential and two-dimensional; (ii) The waves are linear (small amplitude to wavelength ratio); (iii) The Froude number of the background current based on the source discharge and depth is small; (iv) Wavelength to source depth ratio is of the order of one, and (v) The wave potential is regular in the whole flow domain. The main result is the appearance of reflected waves, including an estimation for the reflection coefficient.

NOTATION

a — amplitude
$a_l^{(\pm)}$, $l = 1,2,3,4$ — constants (Eq. (A.2))
a_1, a_2, a_{11}, a_{12}, a_{22} — coefficients depending on the current velocity field
A, B, C, D — unknown constants
$c_l^{(\pm)}$, $l = 1,2,3,4$ — constants (Eq. (A.2))
e_1, e_2 — unknown constants
E — coefficient depending on the background current
E_i — exponential integral
$F_1, F_2, \bar{F}_1, \bar{F}_2$ — auxiliary potentials
$\mathbf{F}$ — a two-component F_1, F_2 column vector
${}_2F_1$ — Gauss' hypergeometric function
$\mathcal{F}$ — auxiliary 2×2 matrix
g — acceleration of gravity
$g^{(\pm)}$ — integrals, Eqs. (3.13a, b)
h — depth of source beneath the free-water surface
$h^{(\pm)}$ — integrals, Eqs. (3.15a, b)
H — coefficient depending on the background current
i, j — imaginary units
k — wave number
L — a two-component column vector depending on T, T_1, T_2
$L_{3,4}^{(-)}$ — integration paths
$\mathbf{M}$ — 2×2 matrix depending on T
N — the overall elevation of the free surface
R_c — reflection coefficient
s — variable of integration
t — time
T, T_1, T_2 — auxiliary functions, Eqs. (3.7), (3.8a, b)
T_c — transmission coefficient
x — horizontal coordinate
y — vertical positive upwards coordinate
z — $x + iy$ — complex variable

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α	— small parameter characterizing amplitude to wavelength ratio
β	— characteristic Froude number of the source
$\delta^{(+)}$	— a constant
η	— the wave profile
$\eta^{(0)}$	— free-surface elevation of the background current
η_{local},	— η_{global} — see Eqs. (4.3), (4.4), (4.7)
λ	— wavelength of incoming wave
ξ	— integration variable
τ	— integration variable
Φ	— overall velocity potential
$\phi^{(0)}$	— the background current potential
ϕ	— the wave potential
$\phi^{(R)}, \phi^{(J)}$	— real and imaginary parts of the wave potential, respectively
ψ	— current potential, Eq. (2.12)
Ψ	— complex current potential
ω	— frequency of incoming waves

1. INTRODUCTION

Knowledge about the mechanism of water waves propagating on nonuniform background currents is of importance in various fields of science and engineering. In specific connection with the present work we mention the hydraulic-breakwater, which is a device used to decrease wave height by creating currents opposed to the direction of wave propagation. Although the hydraulic-breakwater has been investigated theoretically and experimentally by many authors, it still appears that the exact physical phenomenon is not clear enough (*Nece, Richey and Seetlarama Rao*, 1968).

Most studies dealing with wave current interaction (*Phillips*, 1966, *p.* 56; *Whitham*, 1974, *p.* 564) consider short waves (in comparison to a typical length scale of the current) which are refracted due to the nonuniformity of the current. In the two-dimensional problem the refraction manifests itself in amplitude and wavelength changes. The present article discusses linear waves passing over a potential source whose depth beneath the water surface (which represents the length scale of the background current) is of the same order of magnitude as the wavelength (similar to what occurs in the hydraulic-breakwater case). The aim of this study is to show that, in addition to refraction, standing waves in the near field and reflection in the far field may appear in problems similar to that discussed.

2. STATEMENT OF THE PROBLEM

Two-dimensional potential flow with a free surface in water of infinite depth is considered. Variables are made dimensionless with respect to the length scale $(\lambda/2\pi)$ and the corresponding time scale $\omega^{-1} = (\lambda/2\pi g)^{\frac{1}{2}}$, where λ, ω are the wavelength and frequency of the incoming waves, respectively. The velocity potential $\Phi(x, y, t)$ (x —horizontal coordinate, y —vertical positive upwards coordinate and t —the time) is defined in the domain $-\infty < x < \infty$, $-\infty < y < N$ where $y = N(x, t)$ is the equation of the freesurface. $\Phi(x, y, t)$ satisfies the Laplace equation

$$\nabla^2\Phi = \Phi_{,xx} + \Phi_{,yy} = 0 \qquad (y \leqq N), \qquad (2.1)$$

in the flow domain and the free-surface boundary condition

$$\Phi_{,tt} + 2\Phi_{,x}\Phi_{,xt} + 2\Phi_{,y}\Phi_{,yt} + \Phi_{,x}^2\Phi_{,xx} + 2\Phi_{,x}\Phi_{,y}\Phi_{,xy} + \Phi_{,y}^2\Phi_{,yy} + \Phi_{,y} = 0 \qquad (y = N). \qquad (2.2)$$

The unknown function $N(x, t)$ is related to Φ by

$$N = -\Phi_{,t} - (\Phi_{,x}^2 + \Phi_{,y}^2)/2 \qquad (y = N). \qquad (2.3)$$

Equation (2.3) is the Bernoulli equation, while Eq. (2.2) is derived from both kinematic and dynamic conditions. We consider here a potential which is the sum of two basic components: (i) a steady source located beneath the water surface, and (ii) an unsteady term which originates from a simple harmonic incoming wave from $x \to -\infty$. The complicated form of the boundary condition (2.2) and the fact that it has to be satisfied on the free surface, whose form is part of the problem, require two simplifying assumptions in regard to the waves and background current, as follows:

(a) Let (α) characterize the amplitude to wavelength ratio and let

$$\Phi(x,y,t)=\phi^{(0)}(x,y)+\alpha[\phi(x,y)e^{-jt}+\phi^*(x,y)e^{jt}]+O(\alpha^2), \quad (2.4)$$

$$N(x,t)=\eta^{(0)}(x)+\alpha[\eta(x)e^{-jt}+\eta^*(x)e^{jt}]+O(\alpha^2) \quad (2.5)$$

be small amplitude ($\alpha \ll 1$) expansions of the potential and free-surface elevation, respectively. The upper index (0) indicates that the term represents the back-ground current, while the asterisk (*) stands for the complex conjugate value. Substitution of Eqs. (2.4), (2.5) into the exact equations, (2.1), (2.2), (2.3), and asymptotic and Taylor expansions yield at zero order (α^0) the steady current problem

$$\nabla^2\phi^{(0)}=0 \qquad (y \leqq \eta^{(0)}), \quad (2.6)$$

$$(\phi^{(0)}_{,x})^2\phi^{(0)}_{,xx}+2\phi^{(0)}_{,x}\phi^{(0)}_{,y}\phi^{(0)}_{,xy}+(\phi^{(0)}_{,y})^2\phi^{(0)}_{,yy}+\phi^{(0)}_{,y}=0 \qquad (y=\eta^{(0)}), \quad (2.7)$$

$$\eta^{(0)}=-[(\phi^{(0)}_{,x})^2+(\phi^{(0)}_{,y})^2]/2 \qquad (y=\eta^{(0)}), \quad (2.8)$$

and at first order (α^1) the following linear wave problem:

$$\nabla^2\phi=0 \qquad (y \leqq \eta^{(0)}), \quad (2.9)$$

$$a_0\phi+a_1\phi_{,x}+a_2\phi_{,y}+a_{11}\phi_{,xx}+a_{12}\phi_{,xy}+a_{22}\phi_{,yy}=0 \qquad (y=\eta^{(0)}) \quad (2.10)$$

$$\eta=(j\phi-\phi^{(0)}_{,x}\phi_{,x}-\phi^{(0)}_{,y}\phi_{,y})/H \qquad (y=\eta^{(0)}), \quad (2.11)$$

where all the coefficients (given in Appendix A) are dependent on the current velocity field, $H=1+\phi^{(0)}_{,x}\phi^{(0)}_{,xy}+\phi^{(0)}_{,y}\phi^{(0)}_{,yy}>0$ such that Taylor's stability criterion is satisfied for the steady current.

(b) The Froude number (β) of the source, based on the source discharge and the chosen characteristic length ($\lambda/2\pi$), is small ($\beta \ll 1$), so that

$$\phi^{(0)}=\beta\psi(x,y)+O(\beta^2), \quad (2.12)$$

where ψ, $\psi_{,x}$, $\psi_{,y}$, $\cdots$ are of order one (remembering that the source depth and the wave length are of the same order of magnitude). Substitution of Eq. (2.12) into Eqs. (2.6), (2.7), (2.8) and keeping terms up to $O(\beta)$ yields the approximate current problem

$$\nabla^2\psi=0 \qquad (y \leqq 0), \quad (2.13)$$

$$\psi_{,y}=0 \qquad (y=0), \quad (2.14)$$

$$\eta^{(0)}=O(\beta^2). \quad (2.15)$$

Substitution of Eq. (2.12) into the linearized wave problem (2.9), (2.10), (2.11) gives at the same order of precision

$$\nabla^2\phi \qquad (y \leqq 0), \quad (2.16)$$

$$\phi-\phi_{,y}=-j\beta(2\psi_{,x}\phi_{,x}+\psi_{,xx}\phi_{,y}) \qquad (y=0), \quad (2.17)$$

$$\eta=j\phi-\beta\psi_{,x}\phi_{,x} \qquad (y=0). \quad (2.18)$$

As mentioned, we consider a source at depth h beneath the free surface. Hence, the solution of Eqs. (2.13) and (2.14) is

$$\psi=\ln[x^2+(y+h)^2]^{\frac{1}{2}}+\ln[x^2+(y-h)^2]^{\frac{1}{2}}. \quad (2.19)$$

It is emphasized that the difference between the present and previous studies, like Pillips (1966) and Whitham (1974), which discuss short waves, is in the fact that the present typical length scale of the background current (h) is of order one.

The mathematical problem is summarized as follows: A regular potential ϕ satisfying the Laplace equation (2.16) and the mixed boundary condition (2.17) is sought in the lower half-plane. The boundary condition contains variable coefficients which depend upon the source potential (2.19).

3. SOLUTION

Substituting the separation of ϕ ($\phi=\phi^{(R)}+j\phi^{(J)}$) into Eq. (2.16) and the boundary condition (2.17) yields

$$\nabla^2\phi^{(R)}=\nabla^2\phi^{(J)}=0 \qquad (y \leqq 0), \quad (3.1a,b)$$

$$\phi^{(R)}-\phi^{(R)}_{,y}=2\beta\psi_{,x}\phi^{(J)}_{,x}+\beta\psi_{,xx}\phi^{(J)}_{,y} \quad (y=0), \quad (3.2a)$$

$$\phi^{(J)} - \phi^{(J)}_{,y} = -2\beta\psi_{,x}\phi^{(R)}_{,x} - \beta\psi_{,xx}\phi^{(R)}_{,y} \qquad (y=0). \tag{3.2b}$$

Define $\phi^{(R)} = \mathrm{Re}_i\{F_1(z)\}$, $\phi^{(J)} = \mathrm{Re}_i\{F_2(z)\}$ and $\beta\psi = \mathrm{Re}_i\{\Psi(z)\}$, where $\Psi(z) = \beta \ln(z+ih) + \beta\ln(z-ih)$, $z = x+iy$, and F_1, F_2 are holomorphic in the lower half-plane satisfying, according to Eqs. (3.2a, b), the following equations:

$$\mathrm{Re}_i\{F_1 - iF_1'\} = \mathrm{Re}_i\{(2\Psi' + i\Psi'')F_2'\} \qquad (y=0), \tag{3.3a}$$

$$\mathrm{Re}_i\{F_2 - iF_2'\} = -\mathrm{Re}_i\{(2\Psi' + i\Psi'')F_1'\} \qquad (y=0). \tag{3.3b}$$

With the aid of $\bar{F}_1 = F_1^*(z^*)$, $\bar{F}_2 = F_2^*(z^*)$, which are holomorphic in the upper half-plane, Eqs. (3.3a, b) become

$$\mathrm{Re}_i\{F_1 - iF_1'\} = \Psi'(F_2' + \bar{F}_2') + (i\Psi''/2)(F_2' - \bar{F}_2') \qquad (y=0), \tag{3.4a}$$

$$\mathrm{Re}_i\{F_2 - iF_2'\} = -\Psi'(F_1' + \bar{F}_1') - (i\Psi''/2)(F_1' - \bar{F}_1') \qquad (y=0). \tag{3,4b}$$

Thus for $(F_1 - iF_1')$ and $(F_2 - iF_2')$ we obtain Dirichlet problems in the lower half-plane whose solutions are given by

$$F_1 - iF_1' = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - z}\cdot[\psi'(F_2' + \bar{F}_2') + (i\Psi''/2)(F_2' - \bar{F}_2')] \qquad (y \leqq 0), \tag{3.5a}$$

$$F_2 - iF_2' = \frac{-i}{\pi}\int_{-\infty}^{\infty} \frac{d\xi}{\xi - z}[\Psi'(F_1' + \bar{F}_1') + (i\Psi''/2)(F_1' - \bar{F}_1')] \qquad (y \leqq 0). \tag{3.5b}$$

In order to perform the above integrations an infinitely large semicircle in the lower half-plane is closed for integrand terms containing F_1', F_2', while for terms with $\bar{F}_1'$, $\bar{F}_2'$ the same is done in the upper half-plane. Assuming F_1', F_2', $\bar{F}_1'$, $\bar{F}_2'$ are bounded, the integrals along the semicircles vanish, and the result, as given by the residues, is

$$F_1 - iF_1' = TF_2' + T_1, \tag{3.6a}$$

$$F_2 - iF_2' = TF_1' + T_2, \tag{3.6b}$$

where

$$T = 2\Psi' + i\Psi'' = \beta[2/(z+ih) + 2/(z-ih) - i/(z+ih)^2 + (-i)/(z-ih)^2], \tag{3.7}$$

$$\begin{aligned} T_1 = \beta\{&[iF_2''(-ih) - 2F_2'(-ih)]/(z+ih) + iF_2'(-ih)/(z+ih)^2 \\ &+ [-iF_2''(-ih) + 2F_2'(-ih)]^*/(z-ih) - [iF_2'(-ih)]^*/(z-ih)^2\}, \end{aligned} \tag{3.8a}$$

$$\begin{aligned} T_2 = -\beta\{&[iF_1''(-ih) - 2F_1'(-ih)]/(z+ih) + iF_1'(-ih)/(z+ih)^2 \\ &+ [-iF_1''(-ih) + 2F_1'(-ih)]^*/(z-ih) - [iF_1'(-ih)]^*/(z-ih)^2\}. \end{aligned} \tag{3.8b}$$

It is worth mentioning that $F_1'(-ih)$, $F_1''(-ih)$, $F_2'(-ih)$, $F_2''(-ih)$ are unknown and must be calculated. In matrix notation the system of equations (3.6a, b) is written in the form

$$\boldsymbol{F}' - \mathbf{M}\cdot\boldsymbol{F} = \boldsymbol{L} \tag{3.9}$$

where

$$\boldsymbol{F}(z) = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}; \quad \mathbf{M} = \frac{-1}{1-T^2}\begin{bmatrix} i & -T \\ T & i \end{bmatrix}; \quad \boldsymbol{L} = \frac{1}{1-T^2}\begin{bmatrix} iT_1 - TT_2 \\ iT_2 + TT_1 \end{bmatrix}.$$

The homogenous solution of the system (3.9) is

$$\mathcal{F}(z)=\begin{bmatrix} \exp\left(-i\int^{z}\frac{ds}{1+T}\right) & \exp\left(-i\int^{z}\frac{ds}{1-T}\right) \\ i\exp\left(-i\int^{z}\frac{ds}{1+T}\right) & -i\exp\left(-i\int^{z}\frac{ds}{1-T}\right) \end{bmatrix} \tag{3.10}$$

while the general solution is given by

$$\boldsymbol{F}=\mathcal{F}\cdot\left\{\int_{-\infty}^{z}\mathcal{F}^{-1}(\xi)\cdot \boldsymbol{L}(\xi)d\xi+\boldsymbol{E}\right\},\qquad \boldsymbol{E}=\frac{1}{2}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{3.11}$$

where e_1, e_2 are constants, and

$$\mathcal{F}^{-1}=\frac{1}{2}\cdot\begin{bmatrix} \exp\left(i\int^{\xi}\frac{ds}{1+T}\right) & -i\exp\left(i\int^{\xi}\frac{ds}{1+T}\right) \\ \exp\left(i\int^{\xi}\frac{ds}{1-T}\right) & i\exp\left(i\int^{\xi}\frac{ds}{1-T}\right) \end{bmatrix}$$

Hence

$$F_1=0.5\left[\exp\left(-i\int^{z}\frac{ds}{1+T}\right)\cdot g^{(+)}+\exp\left(-i\int^{z}\frac{ds}{1-T}\right)\cdot g^{(-)}\right], \tag{3.12a}$$

$$F_2=0.5i\left[\exp\left(-i\int^{z}\frac{ds}{1+T}\right)\cdot g^{(+)}-\exp\left(-i\int^{z}\frac{ds}{1-T}\right)\cdot g^{(-)}\right], \tag{3.12b}$$

where

$$g^{(+)}=\beta\int_{-\infty}^{z}\exp\left(i\int^{\xi}\frac{ds}{1+T}\right)\cdot\left[\frac{A}{\xi+ih}+\frac{B}{(\xi+ih)^2}+\frac{C}{\xi-ih}+\frac{D}{(\xi-ih)^2}\right]\frac{d\xi}{1+T}+e_1, \tag{3.13a}$$

$$g^{(-)}=\beta\int_{-\infty}^{z}\exp\left(i\int^{\xi}\frac{ds}{1-T}\right)\cdot\left[\frac{C^*}{\xi+ih}+\frac{D^*}{(\xi+ih)^2}+\frac{A^*}{\xi-ih}+\frac{B^*}{(\xi-ih)^2}\right]\frac{d\xi}{1-T}+e_2. \tag{3.13b}$$

We emphasize two points: firstly, it is possible to show that $g^{(+)}(z)$, $g^{(-)}(z)$ are regular at $z\pm ih$, and secondly, the unknown constants A, B, C and D are a linear combination of $F_1'(-ih)$, $F_1''(-ih)$, $F_2'(-ih)$ and $F_2''(-ih)$. Integrating (3.13a, b) by parts and substituting the results into (3.12a, b) yields

$$F_1=-0.5i\left\{\beta\left[\frac{A+C^*}{z+ih}+\frac{B+D^*}{(z+ih)^2}+\frac{C+A^*}{z-ih}+\frac{D+B^*}{(z-ih)^2}\right]\right.$$
$$\left.+(h^{(+)}+ie_1)\exp\left(-i\int^{z}\frac{ds}{1+T}\right)+(h^{(-)}+ie_2)\exp\left(-i\int^{z}\frac{ds}{1-T}\right)\right\}, \tag{3.14a}$$

$$F_2=0.5\left\{\beta\left[\frac{A-C^*}{z+ih}+\frac{B-D^*}{(z+ih)^2}+\frac{C-A^*}{z-ih}+\frac{D-B^*}{(z-ih)^2}\right]\right.$$
$$\left.+(h^{(+)}+ie_1)\exp\left(-i\int^{z}\frac{ds}{1+T}\right)-(h^{(-)}+ie_2)\exp\left(-i\int^{z}\frac{ds}{1-T}\right)\right\}, \tag{3.14b}$$

where

$$h^{(+)} = \beta \int_{-\infty}^{z} \exp\left(i\int^{\xi} \frac{ds}{1+T}\right) \cdot \left[\frac{A}{(\xi+ih)^2} + \frac{2B}{(\xi+ih)^3} + \frac{C}{(\xi-ih)^2} + \frac{2D}{(\xi-ih)^3}\right] d\xi, \tag{3.15a}$$

$$h^{(-)} = \beta \int_{-\infty}^{z} \exp\left(i\int^{\xi} \frac{ds}{1-T}\right) \cdot \left[\frac{C^*}{(\xi+ih)^2} + \frac{2D^*}{(\xi+ih)^3} + \frac{A^*}{(\xi-ih)^2} + \frac{2B^*}{(\xi-ih)^3}\right] d\xi. \tag{3.15b}$$

Thus going back from F_1, F_2 (3.14a, b) to $\phi^{(R)}$, $\phi^{(J)}$, and finally to ϕ, we obtain

$$\phi = 0.5j\left[(h^{+} + je_1)\exp\left(-j\int^{z}\frac{ds}{1+T}\right)\right]^* - 0.5j(h^{(-)} + je_2)\exp\left(-j\int^{z}\frac{ds}{1-T}\right)$$
$$+0.5j\beta\left\{-\left[\frac{C^*}{z+jh} + \frac{D^*}{(z+jh)^2} + \frac{A^*}{z-jh} + \frac{B^*}{(z-jh)^2}\right] + \left[\frac{A}{z+jh} + \frac{B}{(z+jh)^2} + \frac{C}{z-jh} + \frac{D}{(z-jh)^2}\right]\right\}. \tag{3.16}$$

From Eq. (3.16) on, all the complex quantities are based on the complex unit (j), e.g. $z = x + jy$. The solution (3.16) satisfies Laplace equation (2.16) and the boundary condition (2.17) for any set of chosen values for the six unknown constants A, B, C, D, e_1 and e_2. The calculation of these constants is based on the requirement for a regular solution and the radiation conditions.

In order to compute the unknowns we define the constants $a_l^{(\pm)}$, $c_l^{(\pm)}$, $l = 1,2,3,4$, with the aid of the following identities:

$$\frac{1}{1 \pm T} = \frac{(z+jh)^2(z-jh)^2}{\prod_{l=1}^{4}(z - a_l^{(\pm)})} = 1 + \sum_{l=1}^{4} \frac{c_l^{(\pm)}}{z - a_l^{(\pm)}}. \tag{3.17}$$

The values of $a_l^{(\pm)}$, $c_l^{(\pm)}$ up to order β are given in Appendix A. According to (3.17) we write

$$\int^{z} \frac{d\xi}{1 \pm T(\xi)} = z + \prod_{l=1}^{4} c_l^{(\pm)} \ln(z - a_l^{(\pm)}). \tag{3.18}$$

The above formula is made single-valued after defining the branch-cuts, as shown in Fig. 1.

In order to fulfill the requirement for the regularity of the potential (3.16) along the branch-cuts emerging from the points $a_1^{(\pm)}$, $a_2^{(\pm)}$ we set

$$h^{(+)}(a_1^{(+)}) + je_1 = 0, \tag{3.19a}$$

$$h^{(+)}(a_2^{(+)}) + je_1 = 0, \tag{3.19b}$$

$$h^{(-)}(a_1^{(-)}) + je_2 = 0, \tag{3.19c}$$

$$h^{(-)}(a_2^{(-)}) + je_2 = 0. \tag{3.19d}$$

Substitution of the potential (3.16) in Eqs. (2.18) and (2.5) gives

$$N/\alpha = \mathrm{Re}_j\left\{-(h^{(+)} + je_1)\exp\left(-j\int^{x}\frac{ds}{1+T} + jt\right)\right.$$
$$\left. + (h^{(-)} + je_2)\exp\left(-j\int^{x}\frac{ds}{1-T} - jt\right) + j\beta\psi_{,x}\exp\left(-j\int^{x}\frac{ds}{1+T} + jt\right)\right\} + O(\beta^2). \tag{3.20}$$

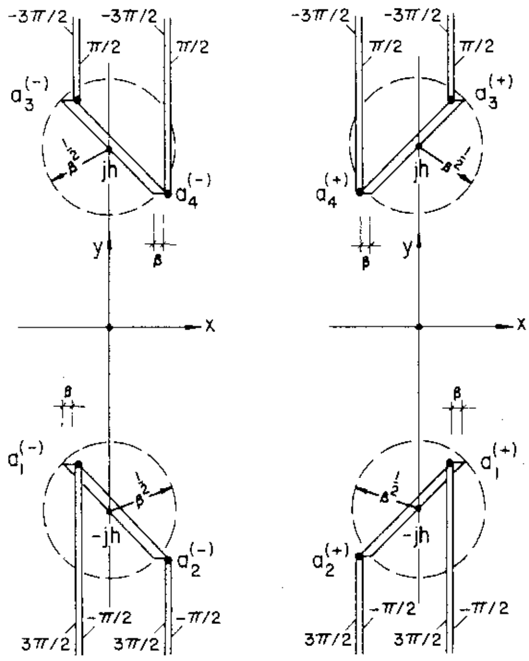

Fig. 1. Branch cuts for expression (3.18).

According to Eqs. (3.18) and (A.2)

$$\lim_{|x|\to\infty}\left(\int^x \frac{ds}{1\pm T(s)}\right) = x \mp 4\cdot\beta\ln|x|. \qquad (3.21)$$

Therefore, from Eq. (3.20) we get

$$N(-\infty)/\alpha = \mathrm{Re}_j\{-je_1\exp[-j(x-t-4\beta\ln|x|)]+je_2\exp[-j(x+t+4\beta\ln|x|)]\}, \qquad (3.22)$$

$$N(+\infty)/\alpha = \mathrm{Re}_j\{-[h^{(+)}(+\infty)+je_1]\cdot\exp[-j(x-t-4\beta\ln|x|)] + [h^{(-)}(+\infty)+je_2]\cdot\exp[-j(x+t+4\beta\ln|x|)]\}. \qquad (3.23)$$

Setting (α) as the incoming wave amplitude in Eq. (3.22) yields

$$e_1 = 1. \qquad (3.24a)$$

From the radiation condition at $x\to+\infty$, taking into consideration Eq. (3.23) we get

$$e_2 = jh^{(-)}(+\infty). \qquad (3.24b)$$

Equations (3.19a, b, c, d) and (3.24a, b) constitute an algebraic linear system of six equations with six unknowns.

An approximate solution of this system (see Appendix B) is

$$A = -3j\exp(-h), \qquad (3.25a)$$

$$B = -\exp(-h), \qquad (3.25b)$$

$$C = 3\beta\{h^{-1}-0.5h^{-2}+h^{-3}/12-2\cdot\exp(-2h)\cdot[j\pi+E_i(2h)]\}\cdot\exp(-h), \qquad (3.25c)$$

$$D = j\beta\{h^{-1}-0.25h^{-2}-2\cdot\exp(-2h)\cdot[j\pi+E_i(2h)]\}\cdot\exp(-h), \qquad (3.25d)$$

while for (e_2) according to (3.24b) the value

$$e_2 = \beta\,(6\pi+0.5j\pi^2)\exp(-2h) \qquad (3.26)$$

is obtained (see Appendix C).

4. RESULTS

Equation (3.16) for the potential, and (3.20) for the wave profile, together with values for the constants given in Eqs. (3.24a), (3.25a, b, c, d), and (3.26) constitute the final solution of the present mathematical problem. According to Eqs. (3.22) and (3.26) the reflection coefficient (R_c), defined as the reflected to incoming wave amplitude ratio, is

$$R_c = |e_2| = 6\pi\beta\,[1+(\pi/12)^2]^{\frac{1}{2}}\exp(-2h)\simeq 20\beta\exp(-2h). \qquad (4.1)$$

Thus, the reflection coefficient is proportional to the source discharge and decreases exponentially with its depth below the water surface (we emphasize that (4.1) is true only for $\beta = o(1)$ and $h = O(1)$). The transmission coefficient (T_c) is given by

$$T_c = |1+jh^{(+)}(+\infty)|. \qquad (4.2)$$

It may be shown that $T_c = 1+O(\beta^2)$, and since the order of approximation of this study is β, it seems impossible to reach a more precise expression for the transmission coefficient.

The wave profile equation (3.20) is represented as a sum of two terms:

$$N = \eta_{\mathrm{local}} + \eta_{\mathrm{global}}; \qquad (4.3)$$

η_{local} is the term representing the incoming wave and the local refractional changes:

$$\eta_{\text{local}} = \alpha \ \mathrm{Re}_j\left\{-j(1-\beta\psi_{,x})\exp\left(-j\int^x \frac{ds}{1+T}+jt\right)\right\}$$
$$\simeq -\alpha(1-2\beta\psi_{,x})\sin\left[\int^x (1-2\beta\psi_{,x})dx - t\right]. \tag{4.4}$$

Thus, neglecting the η_{global} term, the wave amplitude (a) and wave number (k) are given by

$$a/\alpha = 1-2\beta\psi_{,x}, \tag{4.5}$$
$$k = 1-2\beta\psi_{,x}, \tag{4.6}$$

respectively. These expressions are identical with small current velocity expansions of the classical results; see Phillips (1966) and Whitham (1974).

The η_{global} term stands for the global changes as follows:

$$\eta_{\text{global}} = \alpha \mathrm{Re}_j\left\{-h^{(+)}\exp\left(-j\int^x \frac{ds}{1+T}+jt\right)+(h^{(+)}+je_2)\exp\left(-j\int^x\frac{ds}{1-T}-jt\right)\right\}$$
$$\simeq \alpha\mathrm{Re}_j\left\{\frac{-2j\beta\exp(-h)}{x+jh}\exp(jt)\right.$$
$$-\left[4\beta\exp(-h)\int_{-\infty}^x \exp\left(j\int^\xi\frac{ds}{1+T}\right)\frac{d\xi}{\xi+jh}\right]\cdot\exp\left(-j\int^x\frac{ds}{1+T}+jt\right)$$
$$\left.+\left[je_2-2\beta\exp(-h)\int_{-\infty}^x\exp\left(j\int^\xi\frac{ds}{1-T}\right)\frac{d\xi}{\xi-jh}\cdot\exp\left(-j\int^x\frac{ds}{1-T}-jt\right)\right\}. \tag{4.7}$$

The first term in Eq. (4.7) represents standing waves in the near field which decay at $x\to\infty$. The last two terms are rather complicated. Nevertheless, the second one represents waves moving to the right at $|x|\to\infty$ and vanishes for $x\to-\infty$ (as a result of the radiation conditions), while the last term represents waves propagating to the left vanishing at $x\to+\infty$ (3.24b) and describes the reflection at $x\to-\infty$. It is important to mention that the whole global term decays exponentially with h.

Almost all studies on hydraulic breakwaters neglect the η_{global} term (4.7) and base their theoretical considerations solely on refraction (the η_{local} term (4.4)), and wave breaking. Although the submerged source is only an idealized model of an hydraulic breakwater, it seems that the phenomena of standing waves in the near field and some reflection in the far field should be expected and taken into account in actual hydraulic breakwater design.

In summary, the proof of the existence of reflected waves and the development of Eq. (4.1) for estimation of the reflection coefficient are the major achievements of this research.

APPENDIX A

List of Coefficients, Eq. (2.10):

$$\begin{aligned}
a_0 &= -1+jE,\\
a_1 &= -2j\phi^{(0)}_{,x}+2\phi^{(0)}_{,x}\phi^{(0)}_{,xx}+2\phi^{(0)}_{,y}\phi^{(0)}_{,xy}-E\phi^{(0)}_{,x},\\
a_2 &= -2j\phi^{(0)}_{,y}+2\phi^{(0)}_{,x}\phi^{(0)}_{,xy}+2\phi^{(0)}_{,y}\phi^{(0)}_{,yy}-E\phi^{(0)}_{,y}+1,\\
a_{11} &= [\phi^{(0)}_{,x}]^2;\quad a_{12}=2\phi^{(0)}_{,x}\phi^{(0)}_{,y};\quad a_{22}=[\phi^{(0)}_{,y}]^2;\\
E &= \frac{\partial}{\partial y}\{[\phi^{(0)}_{,x}]^2\cdot\phi^{(0)}_{,xx}+2\phi^{(0)}_{,x}\phi^{(0)}_{,y}\phi^{(0)}_{,xy}+[\phi^{(0)}_{,y}]^2\cdot\phi^{(0)}_{,yy}+\phi^{(0)}_{,y}\}/H.
\end{aligned} \tag{A.1}$$

List of Coefficients, Eq. (3.17):

$$
\begin{aligned}
a_1^{(+)} &= -jh + \beta^{\frac{1}{2}}\exp(j\pi/4) - \beta \quad &;\quad c_1^{(+)} &= 0.5\beta^{\frac{1}{2}}\exp(j\pi/4) - \beta,\\
a_2^{(+)} &= -jh - \beta^{\frac{1}{2}}\exp(j\pi/4) - \beta \quad &;\quad c_2^{(+)} &= -0.5\beta^{\frac{1}{2}}\exp(j\pi/4) - \beta,\\
a_3^{(+)} &= jh + \beta^{\frac{1}{2}}\exp(j\pi/4) - \beta \quad &;\quad c_3^{(+)} &= 0.5\beta^{\frac{1}{2}}\exp(j\pi/4) - \beta,\\
a_4^{(+)} &= jh - \beta^{\frac{1}{2}}\exp(j\pi/4) - \beta \quad &;\quad c_4^{(+)} &= -0.5\beta^{\frac{1}{2}}\exp(j\pi/4) - \beta,\\
a_1^{(-)} &= -jh + \beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta \quad &;\quad c_1^{(-)} &= 0.5\beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta,\\
a_2^{(-)} &= -jh - \beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta \quad &;\quad c_2^{(-)} &= -0.5\beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta,\\
a_3^{(-)} &= jh + \beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta \quad &;\quad c_3^{(-)} &= 0.5\beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta,\\
a_4^{(-)} &= jh - \beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta \quad &;\quad c_4^{(-)} &= -0.5\beta^{\frac{1}{2}}\exp(3j\pi/4) + \beta,
\end{aligned}
\tag{A.2}
$$

APPENDIX B

Approximate Solution of the System (19a, b, c, d and 3.24a, b)

After elimination of e_1 by (3.24a) and e_2 by (3.24b) from (3.19a, b, c, d) we obtain

$$\beta\int_{a_1^{(+)}}^{-\infty}\exp\left(j\int^{\xi}\frac{ds}{1+T}\right)\cdot\left[\frac{A}{(\xi+jh)^2}+\frac{2B}{(\xi+jh)^3}+\frac{C}{(\xi-jh)^2}+\frac{2D}{(\xi-jh)^3}\right]d\xi = j, \tag{B.1a}$$

$$\beta\int_{a_2^{(+)}}^{-\infty}\exp\left(j\int^{\xi}\frac{ds}{1+T}\right)\cdot\left[\frac{A}{(\xi+jh)^2}+\frac{2B}{(\xi+jh)^3}+\frac{C}{(\xi-jh)^2}+\frac{2D}{(\xi-jh)^3}\right]d\xi = j, \tag{B.1b}$$

$$\beta\int_{a_1^{(-)}}^{+\infty}\exp\left(j\int^{\xi}\frac{ds}{1-T}\right)\cdot\left[\frac{C^*}{(\xi+jh)^2}+\frac{2D^*}{(\xi+jh)^3}+\frac{A^*}{(\xi-jh)^2}+\frac{2B^*}{(\xi-jh)^3}\right]d\xi = 0, \tag{B.1c}$$

$$\beta\int_{a_2^{(-)}}^{+\infty}\exp\left(j\int^{\xi}\frac{ds}{1-T}\right)\cdot\left[\frac{C^*}{(\xi+jh)^2}+\frac{2D^*}{(\xi+jh)^3}+\frac{A^*}{(\xi-jh)^2}+\frac{2B^*}{(\xi-jh)^3}\right]d\xi = 0. \tag{B.1d}$$

It is convenient to rewrite the system (B.1) in the form

$$\int_{a_2^{(+)}}^{a_1^{(+)}}\exp\left(j\int^{\xi}\frac{ds}{1+T}\right)\cdot\left[\frac{A}{(\xi+jh)^2}+\frac{2B}{(\xi+jh)^3}+\frac{C}{(\xi-jh)^2}+\frac{2D}{(\xi-jh)^3}\right]d\xi = 0, \tag{B.2a}$$

$$\int_{a_2^{(-)}}^{a_1^{(-)}}\exp\left(j\int^{\xi}\frac{ds}{1-T}\right)\cdot\left[\frac{C^*}{(\xi+jh)^2}+\frac{2D^*}{(\xi+jh)^3}+\frac{A^*}{(\xi-jh)^2}+\frac{2B^*}{(\xi-jh)^3}\right]d\xi = 0, \tag{B.2b}$$

$$\int_{a_2^{(+)}}^{-\infty}\exp\left(j\int^{\xi}\frac{ds}{1+T}\right)\cdot\left[\frac{A}{(\xi+jh)^2}+\frac{2B}{(\xi+jh)^3}+\frac{C}{(\xi-jh)^2}+\frac{2D}{(\xi-jh)^3}\right]d\xi = j/\beta, \tag{B.2c}$$

$$\int_{a_2^{(-)}}^{+\infty}\exp\left(j\int^{\xi}\frac{ds}{1-T}\right)\cdot\left[\frac{C^*}{(\xi+jh)^2}+\frac{2D^*}{(\xi+jh)^3}+\frac{A^*}{(\xi-jh)^2}+\frac{2B^*}{(\xi-jh)^3}\right]d\xi = 0. \tag{B.2d}$$

The approximate calculation of the integral in Eq. (B.2a), with the integration path given in Fig. 2, is as follows. Substitution of a new variable of integration $\xi = a_2^{(+)} + 2\beta^{\frac{1}{2}}\exp(j\pi/4)\tau$ into the integrand terms leads to the following approximate expressions:

$$
\begin{aligned}
\xi + jh &= 2\beta^{\frac{1}{2}}\exp(j\pi/4)(\tau - \delta^{(+)}), \quad \delta^{(+)} = 0.5[1 + \beta^{\frac{1}{2}}\exp(-j\pi/4)],\\
\xi - jh &= -2jh + O(\beta^{\frac{1}{2}})
\end{aligned}
$$

and according to Eq. (3.18)

$$\exp\left(j\int^{\xi}\frac{ds}{1+T}\right)=\exp(j\xi)\cdot\prod_{l=1}^{4}(\xi-a_l^{(+)})^{jc_l^{(+)}}\simeq \text{const}\cdot\tau^{jc_2^{(+)}}\cdot(1-\tau)^{jc_1^{(+)}}.$$

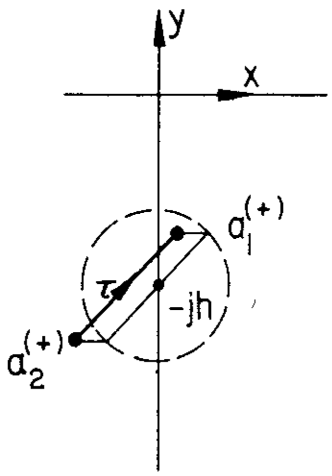

Fig. 2. Integration path for the integral (B.2a).

Using the above terms, Eq. (B.2a) yields

$$\int_0^1 f^{(+)}(\tau)\cdot\tau^{jc_2^{(+)}}\cdot(1-\tau)^{jc_1^{(+)}}\cdot d\tau=0, \qquad \text{(B.3)}$$

where

$$f^{(+)}(\tau)=A/4j\beta(\tau-\delta^{(+)})^2+B/4\beta^{3/2}\exp(3j\pi/4)$$
$$\times(\tau+\delta^{(+)})^3-C/4h^2+D/4jh^3.$$

The result of the four integrals of (B.3) as calculated according to Gradshteyn and Ryshik (1965), p. 285 and p. 287, is

$$\begin{aligned}&A\cdot{}_2F_1(2,1+jc_2^{(+)},2+jc_1^{(+)}+jc_2^{(+)},\delta^{(+)^{-1}})/4\beta j\delta^{(+)^2}\\&-B\cdot{}_2F_1(3,1+jc_2^{(+)},2+jc_1^{(+)}+jc_2^{(+)},\delta^{(+)^{-1}})/4\beta^{3/2}\exp(3j\pi/4)\,\delta^{(+)^3}\\&-C/4h^2+D/4jh^3=0,\end{aligned} \qquad \text{(B.4)}$$

where ${}_2F_1$ is Gauss' hypergeometric function.

Equation (B.2b, c, d) give similar expressions constituting together an algebraic linear system of four equations with four unknowns (A, B, C, D). Considering the whole system, it is concluded that the last two terms in Eq. (B.4) are negligible. Thus,

$$\begin{aligned}B/A=\;&\beta^{\frac{1}{2}}\exp(j\pi/4)\,{}_2F_1(2,1+jc_2^{(+)},2+jc_1^{(+)}+jc_2^{(+)},\delta^{(+)^{-1}})/\\&2\,{}_2F_1(3,1+jc_2^{(+)},2+jc_1^{(+)}+jc_2^{(+)},\delta^{(+)^{-1}}).\end{aligned} \qquad \text{(B.5)}$$

Using the recursive formulae of Gauss (see Gradshteyn and Ryzhik (1965), p. 1044), Eq. (B.5) gives

$$B=-jA/3+O(\beta^{\frac{1}{2}}). \qquad \text{(B.6)}$$

APPENDIX C

The Integral $h^-(+\infty)$

$$h^{(-)}(+\infty)=\beta\int_{-\infty}^{\infty}\exp\left(j\int^{\xi}\frac{ds}{1-T}\right)\cdot\left[\frac{C^*}{(\xi+jh)^2}+\frac{2D^*}{(\xi+jh)^3}+\frac{A^*}{(\xi-jh)^2}+\frac{2B^*}{(\xi-jh)^3}\right]d\xi. \qquad \text{(C.1)}$$

The original integration path (on the real axis) is changed to the paths $L_3^{(-)}$, $L_4^{(-)}$ along the branch-cuts (see Fig. 3), and an upper semicircle whose radius tends to infinity and has no contribution to the integration. Besides, we mention that the residues at $\xi = jh$ are zero.

Taking into consideration Eq. (3.18) and the results (3.25), the approximation for (C.1) is written as

$$h^{(-)}(+\infty)=\beta A^*\int_{-\infty}^{\infty}\exp\,(j\xi)\prod_{i=1}^{4}(\xi-a_i^{(-)})^{jc_i^{(-)}}\cdot\left[\frac{1}{(\xi-jh)^2}+\frac{2j}{3(\xi-jh)^3}\right]d\xi. \qquad \text{(C.2)}$$

Since the major contribution to the integral is from the vicinity of the point $\xi = jh$ and considering the new integration path (Fig. 3), we write

$$h^{(-)}(+\infty)=\beta A^*\exp(-h)\left\{\int_{L_3^{(-)}}d\xi(\xi-a_3^{(-)})^{jc_3^{(-)}}\cdot\left[\frac{1}{(\xi-jh)^2}+\frac{2j}{3(\xi-jh)^3}\right]\right.$$
$$\left.+\int_{L_4^{(-)}}d\xi(\xi-a_4^{(-)})^{jc_4^{(-)}}\cdot\left[\frac{1}{(\xi-jh)^2}+\frac{2j}{3(\xi-jh)^3}\right]\right\}. \qquad \text{(C.3)}$$

Changing the integration variable, $\xi - a_{3,4}^{(-)} = j\tau$, in (C.3) we obtain

$$h^{(-)}(+\infty)=-\beta jA^*\exp(-h)\left\{(2\pi c_3^{(-)}+\pi^2c_3^{(-)2})\cdot\right.$$
$$\cdot\int_0^{\infty}\tau^{jc_3^{(-)}}\cdot\left[\frac{1}{(j\tau+j\beta^{\frac{1}{2}}\exp(j\pi/4)+\beta)^2}+\frac{2j}{3(j\tau+j\beta^{\frac{1}{2}}\exp(j\pi/4)+\beta)^3}\right]d\tau$$
$$+(2\pi c_4^{(-)}+\pi^2c_4^{(-)2})\cdot$$
$$\left.\cdot\int_0^{\infty}\tau^{jc_4^{(-)}}\cdot\left[\frac{1}{(j\tau-j\beta^{\frac{1}{2}}\exp(j\pi/4)+\beta)^2}+\frac{2j}{3(j\tau-j\beta^{\frac{1}{2}}\exp(j\pi/4)\beta)^3}\right]d\tau\right\}. \qquad \text{(C.4)}$$

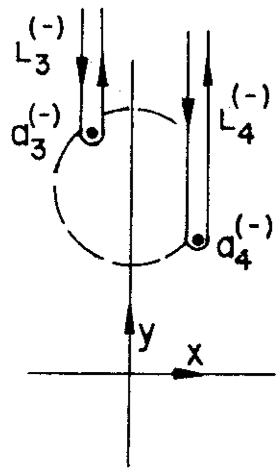

Fig. 3. Integration path for the integral (C.1).

From here, according to Gradshteyn and Ryzhik (1965), p. 285,

$$h^{(-)}(+\infty)=-\beta A^*\exp(-h)\cdot(6\pi+0.5j\pi^2)/3. \qquad \text{(C.5)}$$

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