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DISCRETIZATION OF ZAKHAROV'S EQUATION

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Abstract – In Zakharov's equation the spectral function represents the entire horizontal plane. In practical applications one often has to use a finite number of Fourier-modes that are determined for limited regions of the horizontal plane, but vary from region to region. In this note the Zakharov equation is carefully discretized, and the relationship between the spectral function over the entire wave-number plane and the discrete Fourier-modes is established. The applicability of some special cases of the discretized Zakharov equation is discussed. © Elsevier, Paris

1. Introduction

Zakharov [2] derived an equation describing the slow temporal evolution of the dominant Fourier-components of a weakly nonlinear surface gravity wave-field on deep water. This equation is now known as the Zakharov equation.

In deriving the Zakharov equation the Fourier transform is applied over the entire horizontal plane, resulting in an amplitude spectrum. This spectral function is complex and often quite complicated (i.e. falls within the class of generalized functions). In figure 1a the modulus of the spectral function is schematically represented by the solid line.

In practice however, (i.e. in field or laboratory applications) the Fourier transform is applied only to a limited region of the horizontal plane, resulting in a discrete amplitude spectrum, which varies from region to region. Another reason to consider a discrete amplitude spectrum is that numerical computations are carried out for discrete numbers.

The above mentioned circumstance raises two major questions:

(i) How should one discretize the spectrum?

- and more important:

(ii) How does the discretization affect the governing Zakharov equation?

To answer the above questions, we will first derive a set of governing equations for the slow spatial as well as temporal evolution of a series of discrete Fourier-modes.

As part of the derivation, we establish the relationship between the spectral function (as in figure 1a) and the Fourier-modes (as in figure 1b). The latter result allows us to establish the relation between the Fourier-modes and the free-surface elevation and potential.

The Zakharov equation is presented in section 2 and discretized in section 3. In section 4 we derive relations between the Fourier-modes and the free-surface elevation and potential. Section 5 includes some special cases of the discretized Zakharov equation and a discussion of their range of applicability.

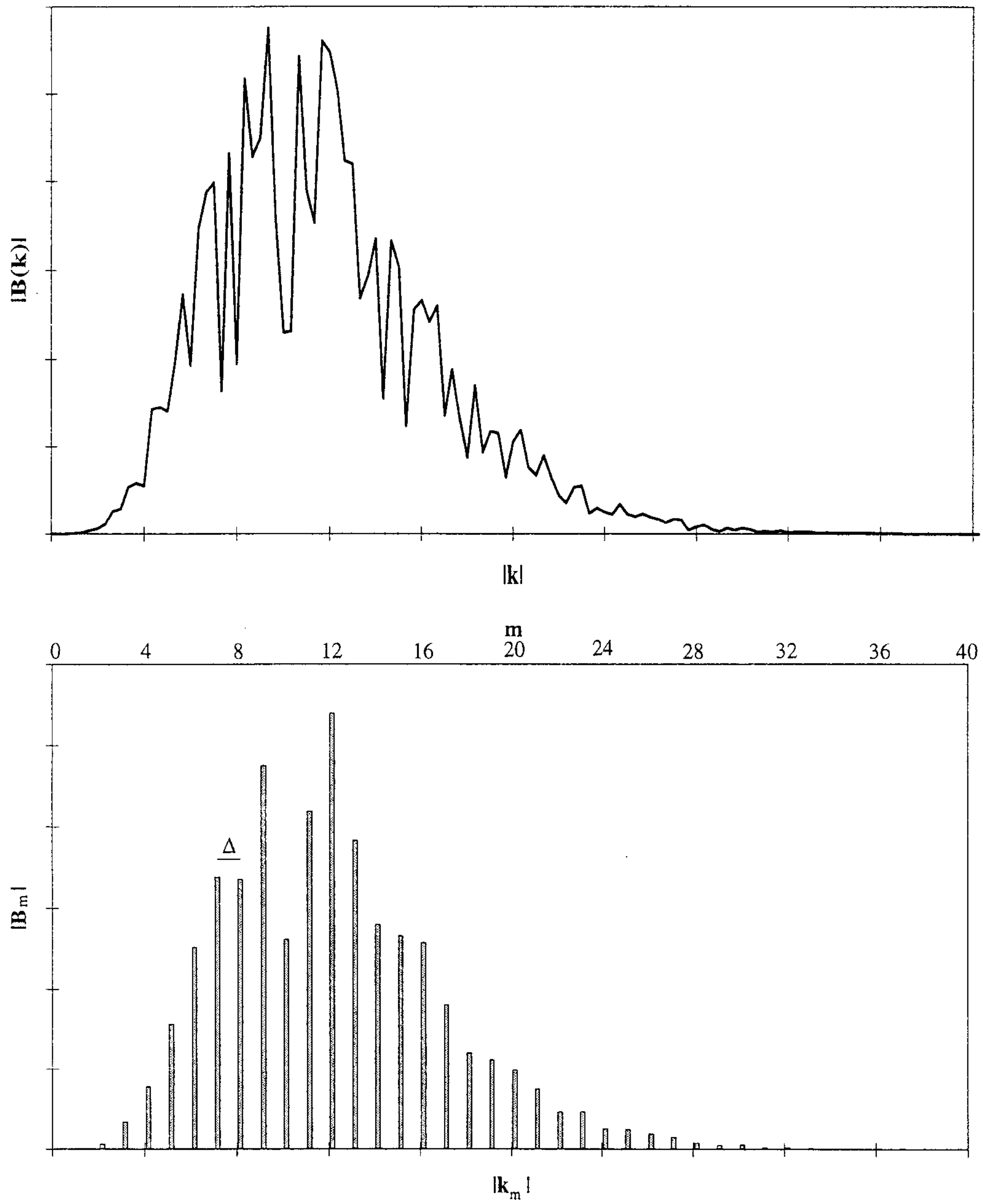

FIGURE 1. a) A schematic drawing of the modulus of the spectral function $|B(\underline{k})|$ for an unidirectional wave-field. b) Fourier-modes of the same wave-field. Note that $|B_m| = f(\underline{k}_m, t,\ and\ \underline{x})$, whereas $|B(\underline{k})| = f(\underline{k}, t)$ only.

2. The Zakharov equation

A generalized complex function is determined from the Fourier transform of the surface elevation, $\hat{\eta}$, and the Fourier transform of the velocity potential at the free surface, $\hat{\psi}$, by

$$\beta(\underline{k}, t) = \left(\sqrt{\frac{g}{2\omega(\underline{k})}}\hat{\eta}(\underline{k}, t) + \mathrm{i}\sqrt{\frac{\omega(\underline{k})}{2g}}\hat{\psi}(\underline{k}, t) \right), \tag{1}$$

where the Fourier transform is given by

$$\hat{f}(\underline{k}) = \frac{1}{2\pi}\int f(\underline{x})\mathrm{e}^{-i\underline{k}\cdot\underline{x}}d\underline{x}. \tag{2}$$

Here $\cdot$ denotes scalar product, $\underline{k} = (k_x, k_y)$ is the wave-number vector, $\underline{x} = (x, y)$ is horizontal space coordinate, and t is time.

The function β is assumed to consist of dominant components B and less dominating components B', ..., such that

$$\beta(\underline{k}, t) = (B(\underline{k}, t) + \epsilon B'(\underline{k}, t) + ...)\,\mathrm{e}^{-i\omega(\underline{k})t}. \tag{3}$$

The slow temporal evolution of the dominant components B of a weakly nonlinear wave-field are governed by Zakharov's equation, cf. [2]

$$\mathrm{i}B_t = \iiint T(\underline{k}, \underline{k}_1, \underline{k}_2, \underline{k}_3)B_1^* B_2 B_3 \delta(\underline{k} + \underline{k}_1 - \underline{k}_2 - \underline{k}_3)\mathrm{e}^{i(\omega+\omega_1-\omega_2-\omega_3)t}d\underline{k}_1 d\underline{k}_2 d\underline{k}_3, \tag{4}$$

where ω is the angular frequency given by the deep-water linear dispersion relation

$$\omega^2 = gk, \tag{5}$$

g being the acceleration due to gravity, k being the length of the wave-number vector $k = |\underline{k}| = \sqrt{k_x^2 + k_y^2}$, i is the imaginary unit, and the kernel $T(\underline{k}, \underline{k}_1, \underline{k}_2, \underline{k}_3)$ can be found in e.g. [1].

The functions β and B are not well-defined for $\underline{k} = \underline{0}$, which represents a current and a change in the mean water level, suggesting that these should be treated differently. However, the wave-induced current velocity and the wave-induced change in the mean water level are of third order in deep water, so that these effects are negligible.

By using the inverse Fourier transform, given by

$$f(\underline{x}) = \frac{1}{2\pi}\int \hat{f}(\underline{k})\mathrm{e}^{i\underline{k}\cdot\underline{x}}d\underline{k}, \tag{6}$$

the surface elevation η, and the velocity potential at the free surface ψ correct to lowest order can be determined from the spectral function B by

$$\eta(\underline{x}, t) = \frac{1}{2\pi}\int \sqrt{\frac{\omega(\underline{k})}{2g}}\left(B(\underline{k}, t)\mathrm{e}^{i(\underline{k}\cdot\underline{x}-\omega(\underline{k})t)} + B^*(\underline{k}, t)\mathrm{e}^{-i(\underline{k}\cdot\underline{x}-\omega(\underline{k})t)}\right)d\underline{k}, \tag{7}$$

and

$$\psi(\underline{x}, t) = -\frac{\mathrm{i}}{2\pi}\int \sqrt{\frac{g}{2\omega(\underline{k})}}\left(B(\underline{k}, t)\mathrm{e}^{i(\underline{k}\cdot\underline{x}-\omega(\underline{k})t)} - B^*(\underline{k}, t)\mathrm{e}^{-i(\underline{k}\cdot\underline{x}-\omega(\underline{k})t)}\right)d\underline{k}, \tag{8}$$

respectively.

3. Discretization of the Zakharov equation

Substituting

$$B = b\mathrm{e}^{i\omega t}, \tag{9}$$

into the Zakharov equation, equation (4), we find

$$\mathrm{i}(b_t + \mathrm{i}\omega b) = \iiint T(\underline{k}, \underline{k}_1, \underline{k}_2, \underline{k}_3) b_1^* b_2 b_3 \delta(\underline{k} + \underline{k}_1 - \underline{k}_2 - \underline{k}_3) d\underline{k}_1 d\underline{k}_2 d\underline{k}_3. \tag{10}$$

Applying the inverse Fourier transform, equation (6), on the above expression, we obtain

$$\int \mathrm{i}(b_t + \mathrm{i}\omega b) \mathrm{e}^{i\underline{k}\cdot\underline{x}} d\underline{k} = \iiiint T(\underline{k}, \underline{k}_1, \underline{k}_2, \underline{k}_3) b_1^* b_2 b_3 \delta(\underline{k} + \underline{k}_1 - \underline{k}_2 - \underline{k}_3) \mathrm{e}^{i\underline{k}\cdot\underline{x}} d\underline{k} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3, \tag{11}$$

which simplifies to

$$\int \mathrm{i}(b_t + \mathrm{i}\omega b) \mathrm{e}^{i\underline{k}\cdot\underline{x}} d\underline{k} = \iiint T(\underline{k}_3 + \underline{k}_2 - \underline{k}_1, \underline{k}_1, \underline{k}_2, \underline{k}_3) b_1^* b_2 b_3 \mathrm{e}^{i(\underline{k}_2 + \underline{k}_3 - \underline{k}_1)\cdot\underline{x}} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3. \tag{12}$$

Note that in both sides of the above expression the integrals are over the entire wave-number plane without any restrictions imposed by a Dirac δ-function.

The integral

$$\int f(\underline{k}, ...) \mathrm{e}^{i\underline{k}\cdot\underline{x}} d\underline{k}, \tag{13}$$

is now replaced by a sum of a countable number of integrals

$$\sum_{m,n} \int f(\underline{k}, ...) h_\Delta(\underline{k} - \underline{k}_{m,n}) \mathrm{e}^{i\underline{k}\cdot\underline{x}} d\underline{k}. \tag{14}$$

Here

$$\underline{k}_{m,n} = \begin{pmatrix} m\Delta \\ n\Delta \end{pmatrix}. \tag{15}$$

are discrete wave-numbers. m, and n are integers that are not simultaneously zero[1], Δ is the increment of the rectangular mesh in the wave-number plane, and h_Δ is a "window"-function given by

$$h_\Delta(\underline{k} - \underline{k}_{m,n}) = \begin{cases} 1, & |k_x - k_{m,n,x}| < \Delta/2 \; and \; |k_y - k_{m,n,y}| < \Delta/2 \\ 0, & otherwise \end{cases} \tag{16}$$

Not taking into account m and n simultaneously zero, means that we neglect the effects of current and change in the mean water level, as explained in the previous section.

The role of the h_Δ-function in equation (14) is to pick a single square when calculating the integral; the summation ensures that all squares have been taken into account.

Introducing

$$\underline{\kappa} = \underline{k} - \underline{k}_{m,n}, \tag{17}$$

expression (14) can be written as

$$\sum_{m,n} \mathrm{e}^{i\underline{k}_{m,n}\cdot\underline{x}} \int f(\underline{k}_{m,n} + \underline{\kappa}, ...) h_\Delta(\underline{\kappa}) \mathrm{e}^{i\underline{\kappa}\cdot\underline{x}} d\underline{\kappa}. \tag{18}$$

[1]This restriction means that our discretized model is unable to treat extremely long waves and background currents.

Introducing the above into equation (12), we find

$$\sum_{m,n} \mathrm{i}e^{i\underline{k}_{m,n}\cdot\underline{x}} \int (b_t + \mathrm{i}\omega(\underline{k}_{m,n}+\underline{\kappa})b)h_\Delta(\underline{\kappa})e^{i\underline{\kappa}\cdot\underline{x}}d\underline{\kappa} \tag{19}$$
$$= \sum_{m_1,n_1}\sum_{m_2,n_2}\sum_{m_3,n_3} e^{i(\underline{k}_{m_2,n_2}+\underline{k}_{m_3,n_3}-\underline{k}_{m_1,n_1})\cdot\underline{x}} \iiint \left(b_1 h_\Delta(\underline{\kappa}_1)e^{i\underline{\kappa}_1\cdot\underline{x}}\right)^* b_2 h_\Delta(\underline{\kappa}_2)e^{i\underline{\kappa}_2\cdot\underline{x}} b_3 h_\Delta(\underline{\kappa}_3)e^{i\underline{k}_3\cdot\underline{x}}$$
$$T(\underline{k}_{m_3,n_3}+\underline{k}_{m_2,n_2}-\underline{k}_{m_1,n_1}+\underline{\kappa}_3+\underline{\kappa}_2-\underline{\kappa}_1, \underline{k}_{m_1,n_1}+\underline{\kappa}_1, \underline{k}_{m_2,n_2}+\underline{\kappa}_2, \underline{k}_{m_3,n_3}+\underline{\kappa}_3)d\underline{\kappa}_1 d\underline{\kappa}_2 d\underline{\kappa}_3.$$

Due to the h_Δ-functions, $|\underline{\kappa}|$ of interest is generally much smaller than $|\underline{k}_{m,n}|$, and thus the interaction coefficient $T(\underline{k}_{m_3,n_3}+\underline{k}_{m_2,n_2}-\underline{k}_{m_1,n_1}+\underline{\kappa}_3+\underline{\kappa}_2-\underline{\kappa}_1, \underline{k}_{m_1,n_1}+\underline{\kappa}_1, \underline{k}_{m_2,n_2}+\underline{\kappa}_2, \underline{k}_{m_3,n_3}+\underline{\kappa}_3)$ can be approximated by $T(\underline{k}_{m_3,n_3}+\underline{k}_{m_2,n_2}-\underline{k}_{m_1,n_1}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3})$ and moved outside the integration, introducing an error of higher order.

Also due to the h_Δ-functions and that $|\underline{\kappa}|$ of interest being much smaller than $|\underline{k}_{m,n}|$, the angular frequency $\omega(\underline{k}_{m,n}+\underline{\kappa})$ can be replaced by its Taylor expansion

$$\omega(\underline{k}_{m,n}+\underline{\kappa}) = \omega_{m,n} + \underline{c}_{g,m,n}\cdot\underline{\kappa} - \frac{g}{8k_{m,n}\omega_{m,n}}\left(\frac{m^2-2n^2}{m^2+n^2}\kappa_x^2 + \frac{n^2-2m^2}{m^2+n^2}\kappa_y^2 + \frac{6mn}{m^2+n^2}\kappa_x\kappa_y\right) + O(\kappa^3), \tag{20}$$

where $\underline{c}_g$ is the group velocity.

Thus equation (19) can be written as

$$\sum_{m,n} \mathrm{i}e^{i\underline{k}_{m,n}\cdot\underline{x}} \int h_\Delta(\underline{\kappa})e^{i\underline{\kappa}\cdot\underline{x}}$$
$$\left(b_t + \mathrm{i}\left[\omega_{m,n} + \underline{c}_{g,m,n}\cdot\underline{\kappa} - \frac{g}{8k_{m,n}\omega_{m,n}}\left(\frac{m^2-2n^2}{m^2+n^2}\kappa_x^2 + \frac{n^2-2m^2}{m^2+n^2}\kappa_y^2 + \frac{6mn}{m^2+n^2}\kappa_x\kappa_y\right)\right]b\right)d\underline{\kappa}$$
$$= \sum_{m_1,n_1}\sum_{m_2,n_2}\sum_{m_3,n_3} T(\underline{k}_{m_3,n_3}+\underline{k}_{m_2,n_2}-\underline{k}_{m_1,n_1}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3})e^{i(\underline{k}_{m_2,n_2}+\underline{k}_{m_3,n_3}-\underline{k}_{m_1,n_1})\cdot\underline{x}}$$
$$\int \left(b_1 h_\Delta(\underline{\kappa}_1)e^{i\underline{\kappa}_1\cdot\underline{x}}\right)^* d\underline{\kappa}_1 \int b_2 h_\Delta(\underline{\kappa}_2)e^{i\underline{\kappa}_2\cdot\underline{x}}d\underline{\kappa}_2 \int b_3 h_\Delta(\underline{\kappa}_3)e^{i\underline{\kappa}_3\cdot\underline{x}}d\underline{\kappa}_3. \tag{21}$$

Now it is rather natural to define a square averaged variable as

$$b_{m,n} = \frac{1}{\Delta^2}\int b(\underline{k}_{m,n}+\underline{\kappa})h_\Delta(\underline{\kappa})e^{i\underline{\kappa}\cdot\underline{x}}d\underline{\kappa}. \tag{22}$$

Spatial derivatives are easily found to

$$\frac{\partial b_{m,n}}{\partial x} = \frac{1}{\Delta^2}\int \mathrm{i}\kappa_x b(\underline{k}_{m,n}+\underline{\kappa})h_\Delta(\underline{\kappa})e^{i\underline{\kappa}\cdot\underline{x}}d\underline{\kappa}, \tag{23}$$

and similarly for $\frac{\partial}{\partial y}$ and higher order derivatives.

Hence equation (21) reduces to

$$\sum_{m,n} \mathrm{i}\mathrm{e}^{i\underline{k}_{m,n}\cdot\underline{x}} \left(\frac{\partial b_{m,n}}{\partial t} + \mathrm{i}\omega_{m,n} b_{m,n} + \underline{c}_{g,m,n} \cdot \nabla b_{m,n} + \frac{\mathrm{i}g}{8k_{m,n}\omega_{m,n}} \left(\frac{m^2-2n^2}{m^2+n^2}\frac{\partial^2 b_{m,n}}{\partial x^2} + \frac{n^2-2m^2}{m^2+n^2}\frac{\partial^2 b_{m,n}}{\partial y^2} + \frac{6mn}{m^2+n^2}\frac{\partial^2 b_{m,n}}{\partial x \partial y} \right) \right)$$
$$= \Delta^4 \sum_{m_1,n_1} \sum_{m_2,n_2} \sum_{m_3,n_3} T(\underline{k}_{m_3,n_3} + \underline{k}_{m_2,n_2} - \underline{k}_{m_1,n_1}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3})$$
$$b^*_{m_1,n_1} b_{m_2,n_2} b_{m_3,n_3} \mathrm{e}^{i(\underline{k}_{m_3,n_3}+\underline{k}_{m_2,n_2}-\underline{k}_{m_1,n_1})\cdot\underline{x}}. \tag{24}$$

The left as well as the right hand sides of the above expression are complex Fourier series with slowly varying coefficients. The complex exponential-functions of the Fourier series are varying on a fast scale, and their coefficients must match for

$$\underline{k}_{M,N} = \underline{k}_{m_3,n_3} + \underline{k}_{m_2,n_2} - \underline{k}_{m_1,n_1}, \tag{25}$$

where capital indices denote a chosen wave-number.

Thus equation (24) reduces to the following set of partial differential equations

$$\mathrm{i}\frac{\partial b_{M,N}}{\partial t} - \omega_{M,N} b_{M,N} + \mathrm{i}\underline{c}_{g,M,N} \cdot \nabla b_{M,N} - \frac{g}{8k_{M,N}\omega_{M,N}} \left(\frac{M^2-2N^2}{M^2+N^2}\frac{\partial^2 b_{M,N}}{\partial x^2} + \frac{N^2-2M^2}{M^2+N^2}\frac{\partial^2 b_{M,N}}{\partial y^2} + \frac{6MN}{M^2+N^2}\frac{\partial^2 b_{M,N}}{\partial x \partial y} \right)$$
$$= \Delta^4 \sum_{m_1,n_1} \sum_{m_2,n_2} \sum_{m_3,n_3} T(\underline{k}_{M,N}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3}) b^*_{m_1,n_1} b_{m_2,n_2} b_{m_3,n_3}$$
$$\delta_K(\underline{k}_{M,N} + \underline{k}_{m_1,n_1} - \underline{k}_{m_2,n_2} - \underline{k}_{m_3,n_3}), \tag{26}$$

where $\delta_K(..)$ denotes Kroneckers δ.

Substituting

$$b_{M,N} = B_{M,N}\mathrm{e}^{-i\omega_{M,N}t}, \tag{27}$$

into the above expression, we finally find

$$\mathrm{i}\frac{\partial B_{M,N}}{\partial t} + \mathrm{i}\underline{c}_g \cdot \nabla B_{M,N} - \frac{g}{8k_{M,N}\omega_{M,N}} \left(\frac{M^2-2N^2}{M^2+N^2}\frac{\partial^2 B_{M,N}}{\partial x^2} + \frac{N^2-2M^2}{M^2+N^2}\frac{\partial^2 B_{M,N}}{\partial y^2} + \frac{6MN}{M^2+N^2}\frac{\partial^2 B_{M,N}}{\partial x \partial y} \right)$$
$$= \Delta^4 \sum_{m_1,n_1} \sum_{m_2,n_2} \sum_{m_3,n_3} T(\underline{k}_{M,N}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3}) B^*_{m_1,n_1} B_{m_2,n_2} B_{m_3,n_3}$$
$$\delta_K(\underline{k}_{M,N} + \underline{k}_{m_1,n_1} - \underline{k}_{m_2,n_2} - \underline{k}_{m_3,n_3}) \mathrm{e}^{i(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3})t}, \tag{28}$$

which is the main result.

In the sequel we call the two first terms on the left hand side "convective terms", the other terms on the left hand side "dispersive terms", and the terms on the right hand side the "nonlinear terms."

The above equation is valid for all pairs of (M, N) and its left hand side has exactly the same structure as the nonlinear Schrödinger equation.

Combining equations (9), (22), and (27), we find that the Fourier-modes $B_{m,n}$ are related to the spectral function $B(\underline{k})$ through

$$B_{m,n} = \frac{1}{\Delta^2} \int B(\underline{k}_{m,n} + \kappa) h_\Delta(\underline{\kappa}) e^{i(\underline{\kappa}\cdot\underline{x} - (\omega(\underline{k}_{m,n}+\underline{\kappa}) - \omega_{m,n})t)} d\underline{\kappa}. \tag{29}$$

4. The relationship between the Fourier-modes and the free surface variables

The relations between the spectral function and the surface elevation and the velocity potential at the free surface as well as the relation between the spectral function and the Fourier-modes are given by equations (7), (8), and (29). It is, however, of practical importance to know the relations between the Fourier-modes and the surface elevation and the velocity potential at the free surface, directly.

To this end we first use equation (9), to find that equations (7), and (8) can be written as

$$\eta(\underline{x}, t) = \frac{1}{2\pi} \int \sqrt{\frac{\omega(\underline{k})}{2g}} \left(b(\underline{k}, t) e^{i\underline{k}\cdot\underline{x}} + b^*(\underline{k}, t) e^{-i\underline{k}\cdot\underline{x}}\right) d\underline{k}, \tag{30}$$

and

$$\psi(\underline{x}, t) = -\frac{\mathrm{i}}{2\pi} \int \sqrt{\frac{g}{2\omega(\underline{k})}} \left(b(\underline{k}, t) e^{i\underline{k}\cdot\underline{x}} - b^*(\underline{k}, t) e^{-i\underline{k}\cdot\underline{x}}\right) d\underline{k}, \tag{31}$$

respectively.

The surface elevation η, and the velocity potential at the free surface ψ correct to lowest order are derived by applying the above discretization technique on equations (30) and (31). First the integrals are written as a sum of a countable number of integrals, i.e. we apply equation (18), resulting in

$$\begin{aligned} \eta(\underline{x}, t) &= \frac{1}{2\pi} \sum_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}} \int \sqrt{\frac{\omega(\underline{k}_{m,n} + \underline{\kappa})}{2g}} \left(b(\underline{k}, t) h_\Delta(\underline{\kappa}) e^{i\underline{\kappa}\cdot\underline{x}}\right) d\underline{\kappa} \\ &+ \frac{1}{2\pi} \sum_{m,n} e^{-i\underline{k}_{m,n}\cdot\underline{x}} \int \sqrt{\frac{\omega(\underline{k}_{m,n} + \underline{\kappa})}{2g}} \left(b(\underline{k}, t) h_\Delta(\underline{\kappa}) e^{i\underline{\kappa}\cdot\underline{x}}\right)^* d\underline{\kappa}, \end{aligned} \tag{32}$$

and

$$\begin{aligned} \psi(\underline{x}, t) &= \frac{\mathrm{i}}{2\pi} \sum_{m,n} e^{-i\underline{k}_{m,n}\cdot\underline{x}} \int \sqrt{\frac{g}{2\omega(\underline{k}_{m,n} + \underline{\kappa})}} \left(b(\underline{k}, t) h_\Delta(\underline{\kappa}) e^{i\underline{\kappa}\cdot\underline{x}}\right)^* d\underline{\kappa} \\ &- \frac{\mathrm{i}}{2\pi} \sum_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}} \int \sqrt{\frac{g}{2\omega(\underline{k}_{m,n} + \underline{\kappa})}} \left(b(\underline{k}, t) h_\Delta(\underline{\kappa}) e^{i\underline{\kappa}\cdot\underline{x}}\right) d\underline{\kappa}. \end{aligned} \tag{33}$$

Due to the h_Δ-functions, $|\underline{\kappa}|$ of interest is generally much smaller than $|\underline{k}_{m,n}|$, and thus the factors of the form $\sqrt{\frac{\omega(\underline{k}_{m,n}+\underline{\kappa})}{2g}}$ can be replaced the leading term in their Taylor expansions

$$\sqrt{\frac{\omega(\underline{k}_{m,n} + \underline{\kappa})}{2g}} = \sqrt{\frac{\omega_{m,n}}{2g}} + O(\kappa), \tag{34}$$

and so on.

By moving out constant terms from the integrations and introducing our new variable $b_{m,n}$, (equation 22), equations (32), and (33) simplify to

$$\eta(\underline{x}, t) = \frac{\Delta^2}{2\pi} \sum_{m,n} \sqrt{\frac{\omega_{m,n}}{2g}} \left(b_{m,n} e^{i\underline{k}_{m,n} \cdot \underline{x}} + b^*_{m,n} e^{-i\underline{k}_{m,n} \cdot \underline{x}} \right), \tag{35}$$

and

$$\psi(\underline{x}, t) = -\frac{i\Delta^2}{2\pi} \sum_{m,n} \sqrt{\frac{g}{2\omega_{m,n}}} \left(b_{m,n} e^{i\underline{k}_{m,n} \cdot \underline{x}} - b^*_{m,n} e^{-i\underline{k}_{m,n} \cdot \underline{x}} \right), \tag{36}$$

respectively.

Using equation (27), we finally find

$$\eta(\underline{x}, t) = \frac{\Delta^2}{2\pi} \sum_{m,n} \sqrt{\frac{\omega_{m,n}}{2g}} \left(B_{m,n} e^{i(\underline{k}_{m,n} \cdot \underline{x} - \omega_{m,n} t)} + B^*_{m,n} e^{-i(\underline{k}_{m,n} \cdot \underline{x} - \omega_{m,n} t)} \right), \tag{37}$$

and

$$\psi(\underline{x}, t) = -\frac{i\Delta^2}{2\pi} \sum_{m,n} \sqrt{\frac{g}{2\omega_{m,n}}} \left(B_{m,n} e^{i(\underline{k}_{m,n} \cdot \underline{x} - \omega_{m,n} t)} - B^*_{m,n} e^{-i(\underline{k}_{m,n} \cdot \underline{x} - \omega_{m,n} t)} \right), \tag{38}$$

respectively.

The opposite relation is equally important and is derived in the following way.

Given the free surface variables η, and ψ in a square with center $\underline{x}_0 = (x_0, y_0)$ and side length $2L$, these can be represented by Fourier series expansions

$$\eta(\underline{x}, t) = \sum_{m,n} H_{m,n} e^{i(mx+ny)\pi/L}, \tag{39}$$

where

$$H_{m,n} = \frac{1}{4L^2} \int_{x_0-L}^{x_0+L} \int_{y_0-L}^{y_0+L} \eta(\underline{x}, t) e^{-i(mx+ny)\pi/L} dy dx, \tag{40}$$

and

$$\psi(\underline{x}, t) = \sum_{m,n} \Psi_{m,n} e^{i(mx+ny)\pi/L}, \tag{41}$$

where

$$\Psi_{m,n} = \frac{1}{4L^2} \int_{x_0-L}^{x_0+L} \int_{y_0-L}^{y_0+L} \psi(\underline{x}, t) e^{-i(mx+ny)\pi/L} dy dx, \tag{42}$$

respectively.

Relating the wave-number resolution parameter Δ to the side-length of the square $2L$ by

$$\Delta = \frac{\pi}{L}, \tag{43}$$

equations (39), and (41) simplify to

$$\eta(\underline{x},t) = \sum_{m,n} H_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}}, \tag{44}$$

$$\psi(\underline{x},t) = \sum_{m,n} \Psi_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}}, \tag{45}$$

where $\underline{k}_{m,n}$ are discrete wave-numbers given by equation (15), whereas

$$H_{m,n} = \frac{\Delta^2}{4\pi^2}\int_A \eta(\underline{x},t) e^{-i\underline{k}_{m,n}\cdot\underline{x}} d\underline{x}, \tag{46}$$

and

$$\Psi_{m,n} = \frac{\Delta^2}{4\pi^2}\int_A \psi(\underline{x},t) e^{-i\underline{k}_{m,n}\cdot\underline{x}} d\underline{x}, \tag{47}$$

respectively. Here $\int_A ...d\underline{x}$ denotes integration over the above mentioned square.

Comparing equation (37) with equation (44), and equation (38) with equation (45), one finds that

$$H_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}} = \frac{\Delta^2}{2\pi}\sqrt{\frac{\omega_{m,n}}{2g}}\left(B_{m,n} e^{i(\underline{k}_{m,n}\cdot\underline{x}-\omega_{m,n}t)} + B^*_{m,n} e^{-i(\underline{k}_{m,n}\cdot\underline{x}-\omega_{m,n}t)}\right), \tag{48}$$

and

$$\Psi_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}} = -\frac{i\Delta^2}{2\pi}\sqrt{\frac{g}{2\omega_{m,n}}}\left(B_{m,n} e^{i(\underline{k}_{m,n}\cdot\underline{x}-\omega_{m,n}t)} - B^*_{m,n} e^{-i(\underline{k}_{m,n}\cdot\underline{x}-\omega_{m,n}t)}\right), \tag{49}$$

for all combinations of (m,n).

Adding equation (48) multiplied with $\sqrt{\frac{g}{2\omega_{m,n}}}$ and equation (49) multiplied with $i\sqrt{\frac{\omega_{m,n}}{2g}}$, we find

$$\sqrt{\frac{g}{2\omega_{m,n}}} H_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}} + i\sqrt{\frac{\omega_{m,n}}{2g}}\Psi_{m,n} e^{i\underline{k}_{m,n}\cdot\underline{x}} = \frac{\Delta^2}{2\pi} B_{m,n} e^{i(\underline{k}_{m,n}\cdot\underline{x}-\omega_{m,n}t)}. \tag{50}$$

which gives us

$$B_{m,n} = \frac{2\pi}{\Delta^2}\left(\sqrt{\frac{g}{2\omega_{m,n}}} H_{m,n} + i\sqrt{\frac{\omega_{m,n}}{2g}}\Psi_{m,n}\right) e^{i\omega_{m,n}t}. \tag{51}$$

Insertion of equations (46), and (47) finally gives us

$$B_{m,n} = \frac{1}{2}\int_A \left(\sqrt{\frac{g}{2\omega_{m,n}}}\eta(\underline{x},t) + i\sqrt{\frac{\omega_{m,n}}{2g}}\psi(\underline{x},t)\right) e^{-i(\underline{k}_{m,n}\cdot\underline{x}-\omega_{m,n}t)} d\underline{x}. \tag{52}$$

5. Some special cases of the discretized Zakharov equation

We define the wave-number resolution parameter, δ, as the ratio between Δ and, say, the spectral peak wave-number k_p

$$\delta = \frac{\Delta}{k_p}. \tag{53}$$

The nonlinearity measure ϵ is estimated by

$$\epsilon = k_p a, \tag{54}$$

where a is a typical amplitude of the wave-field.

Both of the above parameters have to be small; $\epsilon \ll 1$, $\delta \ll 1$. The nonlinearity parameter ϵ because waves rarely reach beyond $ka = 0.25$, since the maximum steepness is limited by breaking. The wave-number resolution parameter δ because of the Fourier approach, i.e. the Fourier analysis has to be applied on a length-scale corresponding to a "large" number of peak wave-lengths.

From equation (28) it is seen that the nonlinear terms are of order $O(\epsilon^3)$, and from equation (21) it is seen that the dispersive terms are of order $O(\epsilon\delta^2)$, and that the gradient term is of order $O(\epsilon\delta)$. The order of the time-derivative is either $O(\epsilon^3)$ which comes from the Zakharov equation, or $O(\epsilon\delta)$ which enters from shifting the angular frequency from ω to $\omega_{M,N}$. If the leading order of the time derivative is $\epsilon\delta$ the two convective terms combined are assumed to be of order $O(\epsilon\delta^2)$.

There are three basically different cases depending on how the wave-number resolution parameter δ, and the nonlinearity measure ϵ, compare to each other.

(i) If the wave-number resolution parameter of the spectrum δ is of the same order as the nonlinearity measure ϵ, the convective terms are of order $O(\epsilon^2)$ individually, and of order $O(\epsilon^3)$ if combined, the dispersive terms are also of order $O(\epsilon^3)$, making equation (28) correct to order $O(\epsilon^3)$. For this case one has to use the full equation (28).

(ii) If one takes a dense discretization, i.e.

$$\delta \ll \epsilon, \tag{55}$$

there are two possibilities:

- If the wave-number resolution parameter of the spectrum δ is of order $O(\epsilon^2)$, the convective terms become of order $O(\epsilon^3)$ individually as well as combined, but the dispersive terms become of order $O(\epsilon^5)$. Thus the governing equation describing the wave-field to leading order simplifies to

$$\begin{aligned} \mathrm{i}\frac{\partial B_{M,N}}{\partial t} + \mathrm{i}\underline{c}_g \cdot \nabla B_{M,N} &= \Delta^4 \sum_{m_1,n_1} \sum_{m_2,n_2} \sum_{m_3,n_3} T(\underline{k}_{M,N}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3}) B^*_{m_1,n_1} B_{m_2,n_2} B_{m_3,n_3} \\ &\qquad \delta_K(\underline{k}_{M,N} + \underline{k}_{m_1,n_1} - \underline{k}_{m_2,n_2} - \underline{k}_{m_3,n_3}) \mathrm{e}^{i(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3})t}. \end{aligned} \tag{56}$$

  The above equation is valid for all pairs of (M, N), and is correct to $O(\epsilon^3)$.

- If, however, the wave-number resolution parameter of the spectrum δ is of order higher than $O(\epsilon^2)$, say of order $O(\epsilon^3)$, which corresponds to a very dense discretization, all terms with spatial derivatives become of order higher than $O(\epsilon^3)$. Thus the governing equation describing the wave-field to leading order reduces to

$$\begin{aligned} \mathrm{i}\frac{\partial B_{M,N}}{\partial t} &= \Delta^4 \sum_{m_1,n_1} \sum_{m_2,n_2} \sum_{m_3,n_3} T(\underline{k}_{M,N}, \underline{k}_{m_1,n_1}, \underline{k}_{m_2,n_2}, \underline{k}_{m_3,n_3}) B^*_{m_1,n_1} B_{m_2,n_2} B_{m_3,n_3} \\ &\qquad \delta_K(\underline{k}_{M,N} + \underline{k}_{m_1,n_1} - \underline{k}_{m_2,n_2} - \underline{k}_{m_3,n_3}) \mathrm{e}^{i(\omega_{M,N}+\omega_{m_1,n_1}-\omega_{m_2,n_2}-\omega_{m_3,n_3})t}. \end{aligned} \tag{57}$$

  The above system of equations, which is the "naive" discretization of Zakharov's equation, is valid for all pairs of (M, N), and correct to order $O(\epsilon^3)$.

(iii) Finally, if the waves have extremely mild steepnesses compared to the wave-number resolution parameter, i.e.

$$\delta \gg \epsilon. \tag{58}$$

Assuming that the nonlinear measure, ϵ, is of order $O(\delta^2)$, the convective terms are of order $O(\delta^3) \sim O(\epsilon^{3/2})$ individually, of order $O(\delta^4) \sim O(\epsilon^2)$ in combination, the dispersive terms are of order $O(\delta^4) \sim O(\epsilon^2)$, and the nonlinear terms are of order $O(\epsilon^3) \sim O(\delta^6)$. Thus the governing equation describing the wave-field to leading order simplifies to

$$\mathrm{i}\frac{\partial B_{M,N}}{\partial t} + \mathrm{i}\underline{c}_g \cdot \nabla B_{M,N} - \frac{g}{8k_{M,N}\omega_{M,N}} \cdot \left(\frac{M^2 - 2N^2}{M^2 + N^2} \frac{\partial^2 B_{M,N}}{\partial x^2} + \frac{N^2 - 2M^2}{M^2 + N^2} \frac{\partial^2 B_{M,N}}{\partial y^2} + \frac{6MN}{M^2 + N^2} \frac{\partial^2 B_{M,N}}{\partial x \partial y} \right) = 0. \quad (59)$$

The above equation models the evolution of a linearized wave-field, it is valid for each pair of (M, N) separately, and is correct to order $O(\epsilon^2) \sim O(\delta^4)$.

The four different regimes for which the different evolution equations are valid are illustrated in figure 2.

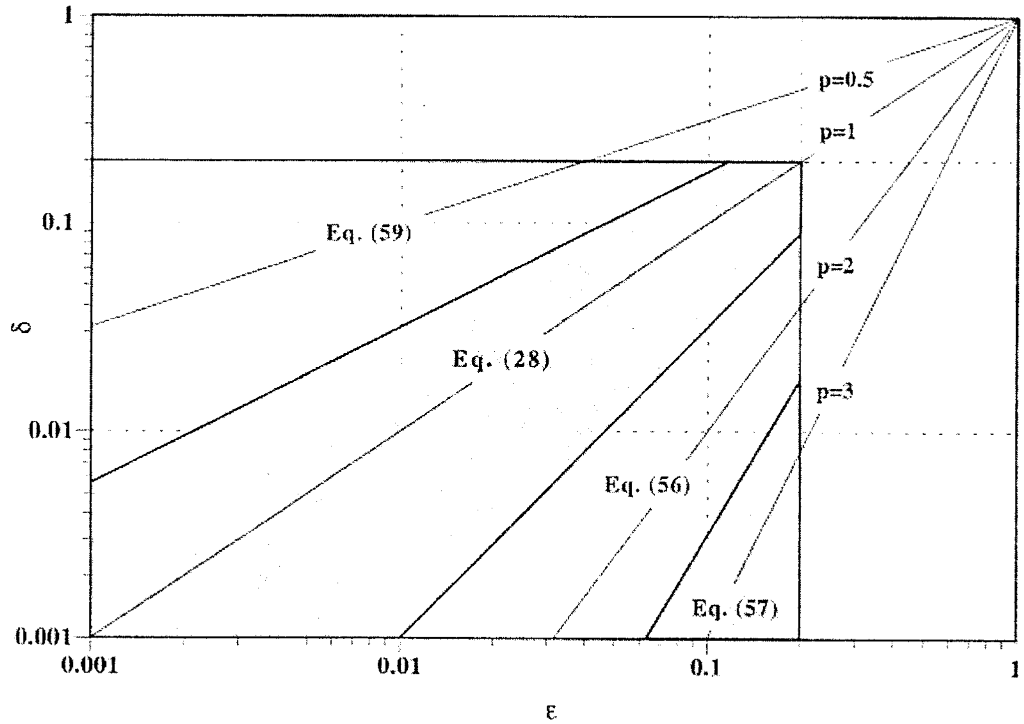

FIGURE 2. Domains of validity of the different equations (indicated on the figure) in the (ϵ, δ)-plane. The diagonal lines are $\delta = \epsilon^p$, where p is also indicated in the figure.

6. Discussion and concluding remarks

We found that the discretized Zakharov equation takes different forms depending on the nonlinearity measure, ϵ, and the wave-number resolution parameter, δ. Four different regimes with corresponding discretized equation were found:

- If the wave-number resolution parameter δ, is of order $O(\epsilon^3)$ or higher, the "naive" discretization of Zakharov's equation, equation (57) holds.
- If the wave-number resolution parameter δ, is of order $O(\epsilon^2)$, equation (56) holds.
- If the wave-number resolution parameter δ, is of order $O(\epsilon)$, the full equation (28) holds.
- If the nonlinearity measure ϵ, is of order $O(\delta^2)$ or higher, equation (59) holds.

The two regimes where equations (28) and (56) hold, are of special interest for a few reasons. They include nonlinear interactions as well as variations in space due to spatial inhomogeneities. This means that these equations are able to model nonlinear wave-wave interactions between modulated wave-trains and wave-packets rather generally. Another reason is that these two regimes together cover a domain in the (ϵ, δ)-plane, which is used for most practical applications. ϵ has typically values in the range from 0.01 to 0.1 and the Fourier analysis is typically applied on a length-scale of 10 to 100 wave-lengths, corresponding to wave-number resolution parameters in the interval from 0.1 to 0.01.

By adding source terms due to wind input and sink terms due to wave-breaking, the discretized Zakharov equation forms a basis for future deterministic wave-forecasting studies.

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