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Water waves in a deep square basin

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The form and evolution of three-dimensional standing waves in deep water are calculated analytically from Zakharov's equation and computationally from the full nonlinear boundary value problem. The water is contained in a basin with a square cross-section, when three-dimensional properties are significant because the natural frequencies of waves in the two directions perpendicular to pairs of sides are the same. It is found that non-periodic standing waves commonly follow forms of cyclic recurrence over long times. The two-dimensional Stokes type of periodic standing waves (dominated by the fundamental harmonic) are shown to be unstable to three-dimensional disturbances, but over long times the waves return cyclically close to their initial state. In contrast, the three-dimensional Stokes type of periodic standing waves are found to be stable to small disturbances. New two-dimensional periodic standing waves with amplitude maxima at other than the fundamental harmonic have been investigated recently (Bryant & Stiassnie 1994). The equivalent three-dimensional standing waves are described here. The new two-dimensional periodic standing waves, like the two-dimensional Stokes standing waves, are found to be unstable to three-dimensional disturbances, and to exhibit cyclic recurrence over long times. Only some of the new three-dimensional periodic standing waves are found to be stable to small disturbances.

1. Introduction

Standing waves may be generated at the free surface of deep water contained between parallel vertical walls. The most important feature is that their spectrum with respect to the coordinate perpendicular to a pair of walls is discrete, with wavenumbers which are integral multiples of π/L, where L is the distance between the walls. If the deep water is contained in a square basin of side L, the wavenumbers in the longitudinal and transverse directions are both integral multiples of π/L.

When waves are generated in a rectangular basin by a wavemaker at one end, cross-waves occur when the frequency of the wavemaker approximates twice one of the resonance frequencies of the transverse standing wave modes and the amplitude of the wavemaker exceeds a certain threshold. This is the phenomenon of parametric resonance, in which energy is transferred from the longitudinal waves to the transverse cross-waves through nonlinear interactions. Miles (1988) presented a theory for such cross-waves on the assumption that their amplitude is slowly modulated in time, and showed how the friction on the walls of the basin could be superimposed on the wave

† See addendum.

evolution as a slow attenuation. The cross-waves are the result of a three-dimensional instability of an otherwise two-dimensional wave. In the present investigation, there is no forcing or friction in order to concentrate on the nonlinear interactions which are the essential part of the cross-wave evolution. The investigation is confined to standing waves, and the basin is chosen to be square to maximize the energy transfer between the longitudinal and transverse waves.

A previous investigation (Bryant & Stiassnie 1994, denoted hereinafter by I) confined attention to nonlinear standing waves in two dimensions, one horizontal and one vertical. The stability and long time evolution were calculated for the Stokes type of standing waves dominated by the first harmonic, and it was shown in particular that the waves can be unstable to sideband modulations, when they evolve into a form of cyclic recurrence. New types of standing waves were investigated for which amplitude maxima occur at harmonics other than the first, under the influence of resonant interactions between these harmonics and the first harmonic. The new standing waves were shown to evolve into different forms of cyclic recurrence. Most results in the investigation were deduced both from Zakharov's equation and from the fully nonlinear boundary value problem.

Verma & Keller (1962) used perturbation expansions in an amplitude parameter to calculate the first nonlinear approximation to three-dimensional standing waves on water of uniform depth in a rectangular basin. Their linear approximation, in the notation of §2 below, is a free wave component of the form

$$a \cos x \cos r^{-1} y \sin \omega t, \tag{1.1}$$

where r is the aspect ratio of the rectangle. Okamura (1984) calculated the regions of instability of two-dimensional standing waves on deep water to both two-dimensional and three-dimensional disturbances. He extended the calculations subsequently (Okamura 1985) to three-dimensional standing waves, using the same linear approximation (1.1) as Verma & Keller (1962). Okamura solved Zakharov's equation, which is valid to the third order of nonlinear interaction, to make calculations up to waves steepnesses of 0.8, compared with wave steepnesses up to 0.15 here. He allowed the disturbance wavenumbers to vary continuously, rather than to take only values which are integral multiples of the fundamental wavenumbers of the basin, and did not include the full range of resonant tertiary interactions that occur when the basin is square.

Nonlinear three-dimensional standing waves in a square basin are constructed in §2 from two-dimensional free wave components parallel to the sides of the basin rather than from fully three-dimensional components such as (1.1). The two formulations are compared in §6, where it is shown that all standing waves constructed from (1.1) in a square basin may be reformulated in terms of the two-dimensional free wave components, but that the reverse is not true.

In this paper, the terminology 'Stokes type' refers to waves for which the amplitudes of spatial harmonics have a 'Stokes ordering', i.e. are monotonically decreasing with order.

2. Nonlinear boundary value problem

The displacement of the water surface is written $z = \eta(x, y, t)$ and the water motion is assumed to be irrotational with the non-dimensional velocity potential $\phi(x, y, z, t)$, where x, y are the two horizontal coordinates, z points vertically upwards from the mean free surface, and t is time. The non-dimensional equations governing the irrotational motion of waves on the free surface of deep water are

$$\nabla^2\phi = 0, \quad z \leqslant \eta(x,y,t), \tag{2.1a}$$

with

$$\eta_t - \phi_z + \eta_x\phi_x + \eta_y\phi_y = 0 \quad \text{on} \quad z = \eta(x,y,t), \tag{2.1b}$$

$$\phi_t + \eta + \tfrac{1}{2}(\phi_x^2+\phi_y^2+\phi_z^2) = 0 \quad \text{on} \quad z = \eta(x,y,t), \tag{2.1c}$$

and

$$|\nabla\phi| \to 0, \quad z \to -\infty. \tag{2.1d}$$

Lengths are made non-dimensional above with respect to L/π, so that the standing wave motion takes place between the vertical planes $x = 0$ and $x = \pi$, $y = 0$ and $y = \pi$. Times are made non-dimensional with respect to the inverse of the lowest linear frequency $(\pi g/L)^{1/2}$ so that the dimensionless linear frequency equals unity. The side conditions are $\phi_x = 0$ on $x = 0$ and $x = \pi$, and $\phi_y = 0$ on $y = 0$ and $y = \pi$.

One of the simplest nonlinear solutions is that for three-dimensional pure standing waves, that is, standing waves which are periodic in time. They consist of free wave components such as

$$a_{101}\cos x\cos t, \quad a_{011}\cos y\cos t, \quad a_{402}\cos 4x\cos 2t, \quad a_{042}\cos 4y\cos 2t, \tag{2.2a}$$

together with their bound wave components. Their Fourier series expansions (before truncation) are

$$\eta = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=l+m \bmod 2}^{\infty} \cos lx\cos my(a_{lmn}\cos n\omega t + b_{lmn}\sin n\omega t), \tag{2.2b}$$

and

$$\phi = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=l+m \bmod 2}^{\infty} \cos lx\cos my\, \mathrm{e}^{(l^2+m^2)^{1/2}z}(c_{lmn}\cos n\omega t + d_{lmn}\sin n\omega t), \tag{2.2c}$$

where $l+m+n$ is even, the coefficients $a_{lmn}, b_{lmn}, c_{lmn}, d_{lmn}$ are constants, and $\omega(\sim 1)$ is the nonlinear frequency of the fundamental harmonic. The constraint that $l+m+n$ is even arises from the values of these parameters in the free components (2.2a), and is associated with the invariance of η and ϕ when x, y and ωt are all changed by π. Following I, the root mean energy

$$\epsilon = \frac{2}{\pi}\left(\int_0^{\pi}\int_0^{\pi}\left(\phi(x,y,\eta,t)\frac{\partial\eta}{\partial t} + \eta^2\right)\mathrm{d}x\,\mathrm{d}y\right)^{1/2} \tag{2.3}$$

is chosen as the measure of the wave amplitude.

It was described in I, §2.2, how wave components in the expansions (2.2b, c) which satisfy the linear dispersion relation

$$n^2 = (l^2+m^2)^{1/2} \tag{2.4}$$

can be excited resonantly and have amplitudes much larger than the amplitudes of other wave components with wavenumbers in their neighbourhood. The properties of the two-dimensional pure standing waves of this type for which $n = 1, 2, 3$ were investigated in I in some detail. The properties of the more general forms of pure standing waves of this type in three dimensions for which $n = 1, 2$ are investigated here.

The fixed point method described in I, §3.2, is used to find computationally the values of the coefficients $a_{lmn}, b_{lmn}, c_{lmn}, d_{lmn}$ needed to satisfy the nonlinear boundary conditions (2.1 b, c) to a given numerical precision (typically 10^{-5}). Also when (2.2b, c) are rewritten

$$\eta = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} a_{lm}(t) \cos lx \cos my, \tag{2.5a}$$

and

$$\phi = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} c_{lm}(t) \cos lx \cos my \, \mathrm{e}^{(l^2+m^2)^{1/2} z}, \tag{2.5b}$$

the time evolution method described in I, §3.3, computes the evolution of non-periodic standing waves. Fourier analysis of the fast time variation of the coefficients a_{lm}, c_{lm} in (2.5a, b), wave period by wave period, enables these expansions to be rewritten

$$\eta = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos lx \cos my (a_{lmn} \cos n\omega t + b_{lmn} \sin n\omega t), \tag{2.6a}$$

and

$$\phi = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos lx \cos my \, \mathrm{e}^{(l^2+m^2)^{1/2} z} (c_{lmn} \cos n\omega t + d_{lmn} \sin n\omega t), \tag{2.6b}$$

where the coefficients $a_{lmn}, b_{lmn}, c_{lmn}, d_{lmn}$ are now functions of slow time.

3. Zakharov's equation

Zakharov's third-order theory (Zakharov 1968) yields an evolution equation in the horizontal Fourier plane

$$\mathrm{i}\frac{\partial B}{\partial t} = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3} B_1^* B_2 B_3 \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3, \tag{3.1}$$

where the new dependent variable $B(\boldsymbol{k}, t)$ is a free component of the wave field. The interpretation of this equation is summarized in I, §2.1, and the kernel $T^{(2)}_{0,1,2,3}$ is given in Stiassnie & Shemer (1984). Note that all cases in the present paper deal with strict resonance conditions, for which $T^{(2)}_{0,1,2,3}$ has all symmetries mentioned by Krasitskii (1994), and the Hamiltonian nature of the problem is maintained.

The wave component for an application to standing waves satisfying (2.4), with non-dimensional wavenumbers

$$\boldsymbol{i}, \boldsymbol{j}, 4\boldsymbol{i}, 4\boldsymbol{j}$$

is written

$$B(\boldsymbol{k}, t) = B_i(t)\,[\delta(\boldsymbol{k}-\boldsymbol{i})+\delta(\boldsymbol{k}+\boldsymbol{i})] + B_j(t)\,[\delta(\boldsymbol{k}-\boldsymbol{j})+\delta(\boldsymbol{k}+\boldsymbol{j})]$$
$$+ B_{4i}(t)\,[\delta(\boldsymbol{k}-4\boldsymbol{i})+\delta(\boldsymbol{k}+4\boldsymbol{i})] + B_{4j}(t)\,[\delta(\boldsymbol{k}-4\boldsymbol{j})+\delta(\boldsymbol{k}+4\boldsymbol{j})], \tag{3.2}$$

where $\boldsymbol{i}, \boldsymbol{j}$ are the unit vectors in the x, y-directions. (It should be noted that the choice of free components with wavenumbers $\boldsymbol{i}, \boldsymbol{j}$ differs from the choice used previously, given in (1.1). The two descriptions are compared in §6.) The dependent variable $B(\boldsymbol{k}, t)$ is replaced by $A(\boldsymbol{k}, t)$ where

$$B_m^2 = \frac{\pi^2}{2m^{1/2}} A_m^2, \quad A_m = a_m + \mathrm{i} m^{1/2} b_m, \tag{3.3}$$

a_m is the complex Fourier amplitude of the wave component with wavenumber $\boldsymbol{m}$, and b_m is the corresponding complex amplitude of the velocity potential on the free surface.

(Compare I, equation (2.14*b*) or Stiassnie & Shemer 1984, equations (2.12*a*, *b*).) Substitution of (3.2) into Zakharov's equation (3.1), (with the superscript (2) omitted), yields

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}B_i}{\mathrm{d}t} = {} & [(T_{i,i,i,i}+2T_{i,-i,i,-i})\,|B_i|^2+2(T_{i,j,i,j}+T_{i,-j,i,-j})\,|B_j|^2]\,B_i \\ & +[2(T_{i,4i,i,4i}+T_{i,-4i,i,-4i})\,|B_{4i}|^2+2(T_{i,4j,i,4j}+T_{i,-4j,i,-4j})\,|B_{4j}|^2]\,B_i \\ & +2T_{i,-i,j,-j}B_i^*\,B_j^2. \end{aligned} \tag{3.4a}$$

Evaluation of the coefficients, combined with the substitution (3.3), reduces (3.4*a*) to

$$\mathrm{i}\frac{\mathrm{d}A_i}{\mathrm{d}t} = -\frac{1}{8}|A_i|^2A_i+\frac{-5+4\surd 2}{56}|A_j|^2A_i+\frac{-19+5\surd 17}{64}|A_{4j}|^2A_i-\frac{3}{16}A_i^*\,A_j^2. \tag{3.4b}$$

The corresponding equation for A_j is

$$\mathrm{i}\frac{\mathrm{d}A_j}{\mathrm{d}t} = -\frac{1}{8}|A_j|^2A_j+\frac{-5+4\surd 2}{56}|A_i|^2\,A_j+\frac{-19+5\surd 17}{64}|A_{4i}|^2\,A_j-\frac{3}{16}A_j^*\,A_i^2. \tag{3.4c}$$

Similarly,

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}B_{4i}}{\mathrm{d}t} = {} & [(T_{4i,4i,4i,4i}+2T_{4i,-4i,4i,-4i})\,|B_{4i}|^2+2(T_{4i,4j,4i,4j}+T_{4i,-4j,4i,-4j})\,|B_{4j}|^2]\,B_{4i} \\ & +[2(T_{4i,i,4i,i}+T_{4i,-i,4i,-i})\,|B_i|^2+2(T_{4i,j,4i,j}+T_{4i,-j,4i,-j})\,|B_j|^2]\,B_{4i}+2T_{4i,-4i,4j,-4j}\,B_{4i}^*\,B_{4j}^2, \end{aligned} \tag{3.5a}$$

which reduces to

$$\mathrm{i}\frac{\mathrm{d}A_{4i}}{\mathrm{d}t} = -4\,|A_{4i}|^2A_{4i}+\frac{4(-5+4\surd 2)}{7}|A_{4j}|^2A_{4i}+\frac{-19+5\surd 17}{32}|A_j|^2A_{4i}-6A_{4i}^*\,A_{4j}^2, \tag{3.5b}$$

and the corresponding equation for A_{4j} is

$$\mathrm{i}\frac{\mathrm{d}A_{4j}}{\mathrm{d}t} = -4\,|A_{4j}|^2A_{4j}+\frac{4(-5+4\surd 2)}{7}|A_{4i}|^2A_{4j}+\frac{-19+5\surd 17}{32}|A_i|^2A_{4j}-6A_{4j}^*\,A_{4i}^2. \tag{3.5c}$$

4. Standing waves of Stokes type

4.1. *Periodic standing waves*

The standing waves of Stokes type are dominated by the fundamental harmonics with wavenumbers i,j, with the higher harmonics decreasing in magnitude in a Stokes ordering. The evolution equations (3.4*b*, *c*) for such waves are

$$\mathrm{i}\frac{\mathrm{d}A_i}{\mathrm{d}t} = -\frac{1}{8}|A_i|^2\,A_i+\frac{-5+4\surd 2}{56}|A_j|^2\,A_i-\frac{3}{16}A_i^*\,A_j^2, \tag{4.1a}$$

$$\mathrm{i}\frac{\mathrm{d}A_j}{\mathrm{d}t} = -\frac{1}{8}|A_j|^2A_j+\frac{-5+4\surd 2}{56}|A_i|^2\,A_j-\frac{3}{16}A_j^*\,A_i^2. \tag{4.1b}$$

Solutions of these equations describing periodic waves are given by

$$A_i = a_i\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}\phi_i},\quad A_j = a_j\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}\phi_j},$$

where $a_i, a_j, \phi_i, \phi_j, \Omega$ are real constants, when (4.1 a, b) may be rewritten

$$\left(\Omega+\tfrac{1}{8}a_i^2-\frac{-5+4\sqrt{2}}{56}a_j^2+\tfrac{3}{16}a_j^2\,\mathrm{e}^{2\mathrm{i}(\phi_j-\phi_i)}\right)a_i=0, \tag{4.2a}$$

$$\left(\Omega+\tfrac{1}{8}a_j^2-\frac{-5+4\sqrt{2}}{56}a_i^2+\tfrac{3}{16}a_i^2\,\mathrm{e}^{2\mathrm{i}(\phi_i-\phi_j)}\right)a_j=0. \tag{4.2b}$$

Equations (4.2 a, b) admit the well-known two-dimensional Stokes standing wave solutions (I, equation (2.14 a))

$$a_j=0, \quad \Omega=-\tfrac{1}{8}a_i^2, \tag{4.3a}$$

oscillating end to end in the x-direction, and

$$a_i=0, \quad \Omega=-\tfrac{1}{8}a_j^2, \tag{4.3b}$$

oscillating end to end in the y-direction. They also admit two families of three-dimensional Stokes standing wave solutions

$$a_j=\pm a_i, \quad \Omega=-\frac{45-8\sqrt{2}}{112}a_i^2=-0.30077a_i^2, \quad \phi_j=\phi_i, \tag{4.3c}$$

and

$$a_j=\pm a_i, \quad \Omega=\frac{-3+8\sqrt{2}}{112}a_i^2=0.07423a_i^2, \quad \phi_j=\phi_i+\frac{\pi}{2}. \tag{4.3d}$$

The standing waves (4.3 c) consist of an oscillation from one corner to the opposite corner with much less motion in the other two corners, and for the standing waves in (4.3 d) an almost flat crest passes around each of the four sides in turn with an almost flat trough on the opposite side.

The Fourier series expansion (2.2 b, c) for the three-dimensional Stokes standing wave solutions are calculated by the fixed point method of I, §3.2, to satisfy the fully nonlinear boundary conditions (2.1 b, c), using the Zakharov solutions (4.3 c, d) as first approximations. The frequency ω, expanded as a polynomial in the fundamental amplitude a_{101} over the range $0<\epsilon<0.15$ (using the NAG least-squares Chebyshev polynomial approximation subroutine E02ADF), is found to have the leading terms

$$\omega=1.0000000-0.30077a_{101}^2+\ldots \tag{4.4a}$$

for the first family, and the leading terms

$$\omega=1.0000000+0.07421a_{101}^2+\ldots \tag{4.4b}$$

for the second family. These are consistent with (4.3 c) and (4.3 d), which gives confidence in the results because the two derivations are independent.

4.2. *Analytical solution of the evolution equations*

When the complex amplitudes in (4.1 a, b) are replaced by the real variables

$$r_1=|A_i|^2, \quad r_2=|A_j|^2, \quad r_3=A_iA_j^*+A_i^*A_j, \quad r_4=(A_iA_j^*-A_i^*A_j)/\mathrm{i}, \tag{4.5}$$

it may be shown that

$$r_{1t}=\tfrac{3}{16}r_3r_4, \quad r_{2t}=-\tfrac{3}{16}r_3r_4, \tag{4.6a, b}$$

$$r_{3t}=-\frac{25+8\sqrt{2}}{112}(r_1-r_2)r_4, \quad r_{4t}=-\frac{17-8\sqrt{2}}{112}(r_1-r_2)r_3, \tag{4.6c, d}$$

where the t-subscript now denotes the derivative. It follows from (4.6a, b) that

$$r_1+r_2 = Q_1, \tag{4.7a}$$

from (4.6a, c) and (4.7a) that

$$r_3^2 = -\frac{2(25+8\sqrt{2})}{21}(r_1^2-Q_1 r_1)+Q_2, \tag{4.7b}$$

and from (4.6a, d) and (4.7a) that

$$r_4^2 = -\frac{2(17-8\sqrt{2})}{21}(r_1^2-Q_1 r_1)+Q_3, \tag{4.7c}$$

where Q_1, Q_2, Q_3 are constants. The sum of (4.7b) and (4.7c), using (4.5) and (4.7a), is

$$r_3^2+r_4^2 = 4r_1 r_2 = 4r_1 r_2+Q_2+Q_3,$$

from which $Q_3 = -Q_2$. Equation (4.6a) can then be rewritten

$$r_{1t}^2 = \frac{(25+8\sqrt{2})(17-8\sqrt{2})}{3136}\left\{-r_1^2+Q_1 r_1+\frac{21Q_2}{2(25+8\sqrt{2})}\right\}\left\{-r_1^2+Q_1 r_1-\frac{21Q_2}{2(17-8\sqrt{2})}\right\}, \tag{4.8}$$

where Q_1, Q_2 are constants determined by the initial conditions in (4.7a, b).

Equation (4.8) is rewritten

$$r_{1t}^2 = C_0\{C_1-(r_1-\tfrac{1}{2}Q_1)^2\}\{C_2-(r_1-\tfrac{1}{2}Q_1)^2\}, \tag{4.9}$$

where

$$C_0 = \frac{(25+8\sqrt{2})(17-8\sqrt{2})}{3136}, \quad C_1 = \tfrac{1}{4}Q_1^2+\frac{21Q_2}{2(25+8\sqrt{2})}, \quad C_2 = \tfrac{1}{4}Q_1^2-\frac{21Q_2}{2(17-8\sqrt{2})}.$$

If $Q_2 > 0$, it can be seen that $C_1 > C_2$, when the substitution $u = (r_1-Q_1/2)/(C_2)^{1/2}$ yields

$$t = \frac{1}{(C_0 C_1)^{1/2}}\int_{u_0}^{u}\frac{du}{\{(1-u^2)(1-mu^2)\}^{1/2}}, \quad m = \frac{C_2}{C_1},$$

$$u = \operatorname{sn}((C_0 C_1)^{1/2}t+F_0),$$

$$\left.\begin{aligned} r_1 &= |A_i|^2 = \tfrac{1}{2}Q_1+(C_2)^{1/2}\operatorname{sn}((C_0 C_1)^{1/2}t+F_0), \\ r_2 &= |A_j|^2 = \tfrac{1}{2}Q_1-(C_2)^{1/2}\operatorname{sn}((C_0 C_1)^{1/2}t+F_0), \end{aligned}\right\} \tag{4.10}$$

where $u = u_0$ at $t = 0$, and

$$F_0 = F(\sin^{-1}(u_0)\,|\,m) = \int_0^{u_0}\frac{du}{\{(1-u^2)(1-mu^2)\}^{1/2}}.$$

If $Q_2 < 0$, then $C_1 < C_2$, and the roles of C_1 and C_2 are reversed in the above calculations. Figure 1 shows graphs of the right-hand side of (4.9) for the three cases, from the top,

$$\text{(i) } Q_2 = 0, \quad C_1 = C_2 = \tfrac{1}{4}Q_1^2, \quad \text{(ii) } Q_2 \neq 0, \quad C_1 C_2 \neq 0, \quad \text{(iii) } Q_2 \neq 0, \quad C_1 C_2 = 0,$$

in which the constant Q_1 is a measure of the total energy and is therefore kept the same for all three curves. The first case is discussed in §4.4, the second case below, and the third case in §4.3.

Phase-plane arguments indicate that the middle curve in figure 1 describes a periodic

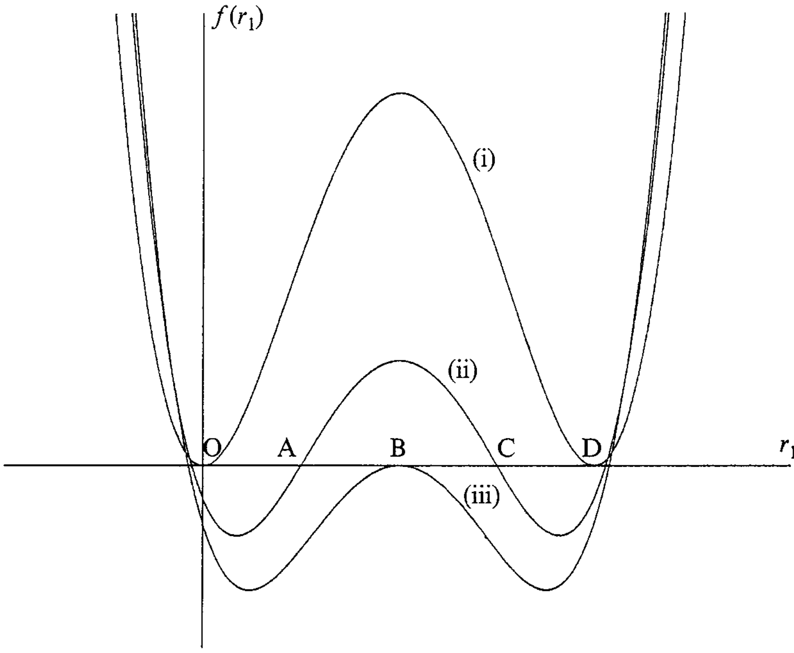

FIGURE 1. The phase-plane diagram for (4.9), $r_{1t}^2 = f(r_1)$, at constant Q_1, for three values of Q_2. (i) $Q_2 = 0$, $C_1 = C_2 = \frac{1}{4}Q_1^2$; (ii) $Q_2 \neq 0$, $C_1 C_2 \neq 0$; (iii) $Q_2 \neq 0$, $C_1 C_2 = 0$.

solution for r_1 between the points A and C, because $r_{1t}^2 = f(r_1) > 0$ in this interval. These are the two points for which one of the bracketed quadratics in (4.8) and (4.9) is zero, which is equivalent to one of (4.7 *b*) and (4.7 *c*) being zero. The solution of (4.9) applicable to the middle curve is stated in (4.10). Also, if

$$A_i = |A_i|\,e^{i\theta_i}, \quad A_j = |A_j|\,e^{i\theta_j}$$

it may be deduced from (4.1 *a, b*) that

$$\theta_{it} = \tfrac{1}{8}Q_1 + \frac{3Q_2}{32r_1}, \quad \theta_{jt} = \tfrac{1}{8}Q_1 + \frac{3Q_2}{32r_2}. \tag{4.11}$$

The fully nonlinear time evolution of a particular example is calculated and compared with the results above, as a check on their validity. The example chosen is a two-dimensional standing wave in the x-direction, (4.3 *a*), with $\epsilon = 0.1$, on which is superimposed at $t = 0$ a two-dimensional standing wave in the y-direction, (4.3 *b*), with $\epsilon = 0.01$, at an initial phase difference of $\pi/4$. The expansions (2.5 *a, b*) are truncated so that they contain all wavenumbers (l, m) such that $l+m \leqslant 5$ with the exception of $(0, 0)$. This gives a system of 40 coefficients a_{lm}, c_{lm}, whose evolution is calculated (I, §3.3) with a local error tolerance of 10^{-11}. A Fourier analysis of the fast time variation of the coefficients produces the expansions (2.6 *a, b*) in which the coefficients have slow time variation.

The fully nonlinear solutions for a_{101}, a_{011}, which are the same as $r_1^{1/2} = |A_i|$, $r_2^{1/2} = |A_j|$ respectively in the Zakharov model, are compared in figure 2(*a, b*) with the corresponding solutions in (4.10) when $A_i = 0.1$, $A_j = 0.01 \exp i\pi/4$ at $t = 0$. The solid curve is the fully nonlinear solution over 5000 wave periods, and the crosses at intervals of 50 wave periods are derived from (4.10). The agreement is excellent, even though the numerical calculations leading to the solid curves in figure 2(*a, b*) are independent of the analytical calculations of the crosses in these figures. The excellent agreement gives confidence in the applicability of the Zakharov model to standing waves at the small but finite amplitude of the example, despite the neglect in the model of nonlinear

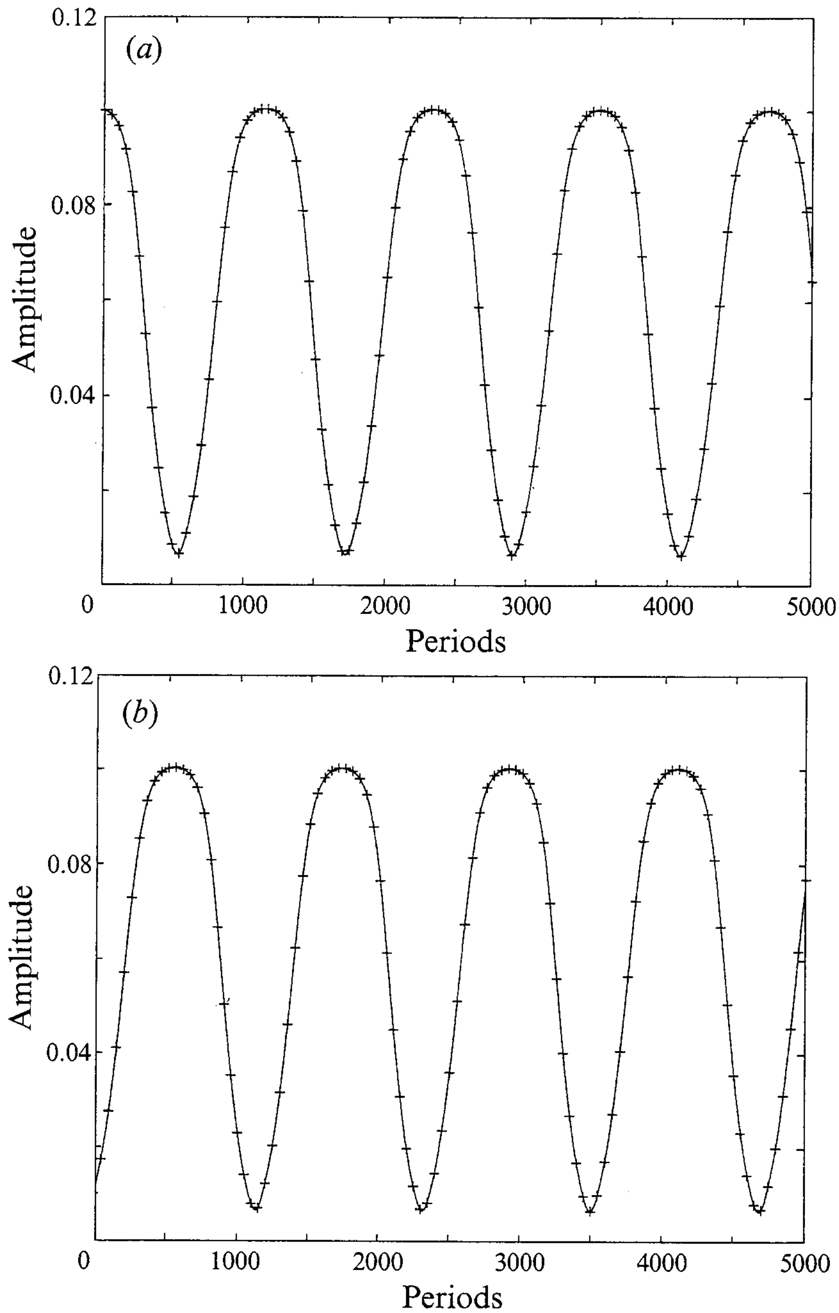

FIGURE 2. A comparison of the fully nonlinear solution for the amplitude of the first harmonic in (*a*) the x-direction and (*b*) the y-direction, drawn as solid curves, with the same amplitude calculated from the Zakharov model, shown as crosses. The initial condition consists of the two-dimensional standing wave of amplitude 0.1 in the x-direction at a phase difference of $\pi/4$ with a standing wave of amplitude 0.01 in the y-direction.

resonant interactions of higher than tertiary order. Figure 2(*a*, *b*) also provides an excellent illustration of a nonlinear periodic interchange of energy between two dominant modes of oscillation, which in this case are the standing waves parallel to each pair of walls of the square cross-section of the basin.

4.3. *Stability of the three-dimensional Stokes standing waves*

The two families of three-dimensional periodic standing waves of Stokes type are given by (4.3*c*) and (4.3*d*). Both families satisfy (4.9) with $r_{1t} = 0$ so that the squares of their moduli lie at the point B of the lowest curve in figure 1. If a small perturbation is applied to the standing waves, their motion is governed by (4.9) with $r_{1t} \neq 0$. The curve describing the right-hand side of (4.9) is then obtained approximately by slightly raising the lowest curve of figure 1, since motion requires $r_{1t}^2 \geqslant 0$, which must therefore have two roots in the neighbourhood of B. The motion described by the curve remains in the neighbourhood of B and the standing waves are therefore stable.

The fully nonlinear time evolution of an example from each family was calculated as a check on the validity of their stability. The examples chosen were both for root-mean energy $\epsilon = 0.1$, integrated over 2000 wave periods, using the same program as that for figure 2. Neither example showed any instability.

4.4. *Three-dimensional instability of two-dimensional Stokes standing waves*

The two families, of two-dimensional periodic standing waves of Stokes type are described by (4.3*a*) and (4.3*b*). Both families satisfy (4.9) with $r_{1t} = 0$, and the squares of their moduli lie at the points D, O respectively of the highest curve in figure 1. If a small three-dimensional perturbation is applied to the standing waves, so that $r_2 > 0$, their motion is governed by (4.9) with $r_{1t} \neq 0$. The curve describing the right-hand side of (4.9) is then obtained approximately by slightly lowering the highest curve of figure 1 since, from (4.7*a*), r_1 lies in the range $0 < r_1 < Q_1$ when $r_2 > 0$, and $r_1 = Q_1$ at the point D.

When the highest curve in figure 1 is slightly lowered, a three-dimensional perturbation of the Stokes standing wave (4.3*a*) is described initially by a point for which $r_{2t} > 0$, $r_{1t} < 0$, (4.7*a*), near but before D. The point moves along the curve with r_1 decreasing, $r_{1t} < 0$, $r_{1t}^2 > 0$, until it reaches the zero of r_{1t} near O. This behaviour is linearly unstable because r_1 moves away from the neighbourhood of D, but r_1 returns cyclically to the neighbourhood of D. The solution is described analytically by (4.10) with the appropriate initial conditions.

The fully nonlinear time evolution of a particular example was calculated and compared with the results above, as a check on their validity. The example chosen is a two-dimensional Stokes standing wave in the x-direction, (4.3*a*), with $\epsilon = 0.1$, which is disturbed at $t = 0$ by a two-dimensional standing wave in the y-direction, (4.3*b*), with $\epsilon = 0.0001$, at an initial phase difference of $\pi/4$. The calculation was made with the same number of harmonics and to the same precision as that described in §4.2.

The fully nonlinear solutions for a_{101}, a_{011}, which are the same as $r_1^{1/2} = |A_i|$, $r_2^{1/2} = |A_j|$ respectively in the Zakharov model, are compared in figure 3(*a*, *b*) with the corresponding solutions in (4.10) when $A_i = 0.1$, $A_j = 0.0001 \exp i\pi/4$ at $t = 0$. The solid curves are the fully nonlinear solutions over 5000 wave periods, and the crosses at intervals of 50 wave periods are derived from (4.10). The agreement is excellent for the first 2000 wave periods, and the subsequent divergence occurs when one of the two components of the first harmonic is close to zero ($\sim 10^{-4}$), although this divergence leaves the form of the evolution correct. The divergence does not occur in figure 2(*a*, *b*), suggesting that some higher-order resonant interactions neglected in the Zakharov model become significant when one of the two components of the first harmonic is close to zero. The figure provides further illustration of the nonlinear periodic interchange of energy between two dominant modes of oscillation.

The exponential growth of $|A_j|$ over the first 600 wave periods of figure 3(*b*), while $|A_i|$ remains almost constant in figure 3(*a*), illustrates the linear instability of the two-dimensional standing wave to the three-dimensional disturbance. The subsequent nonlinear modification of the unstable growth is modelled well by the solution to (4.9) corresponding to a slight lowering of the upper curve in figure 1. Equation (4.9) for the upper curve is

$$r_{1t}^2 = C_0 r_1^2 (Q_1 - r_1)^2,$$

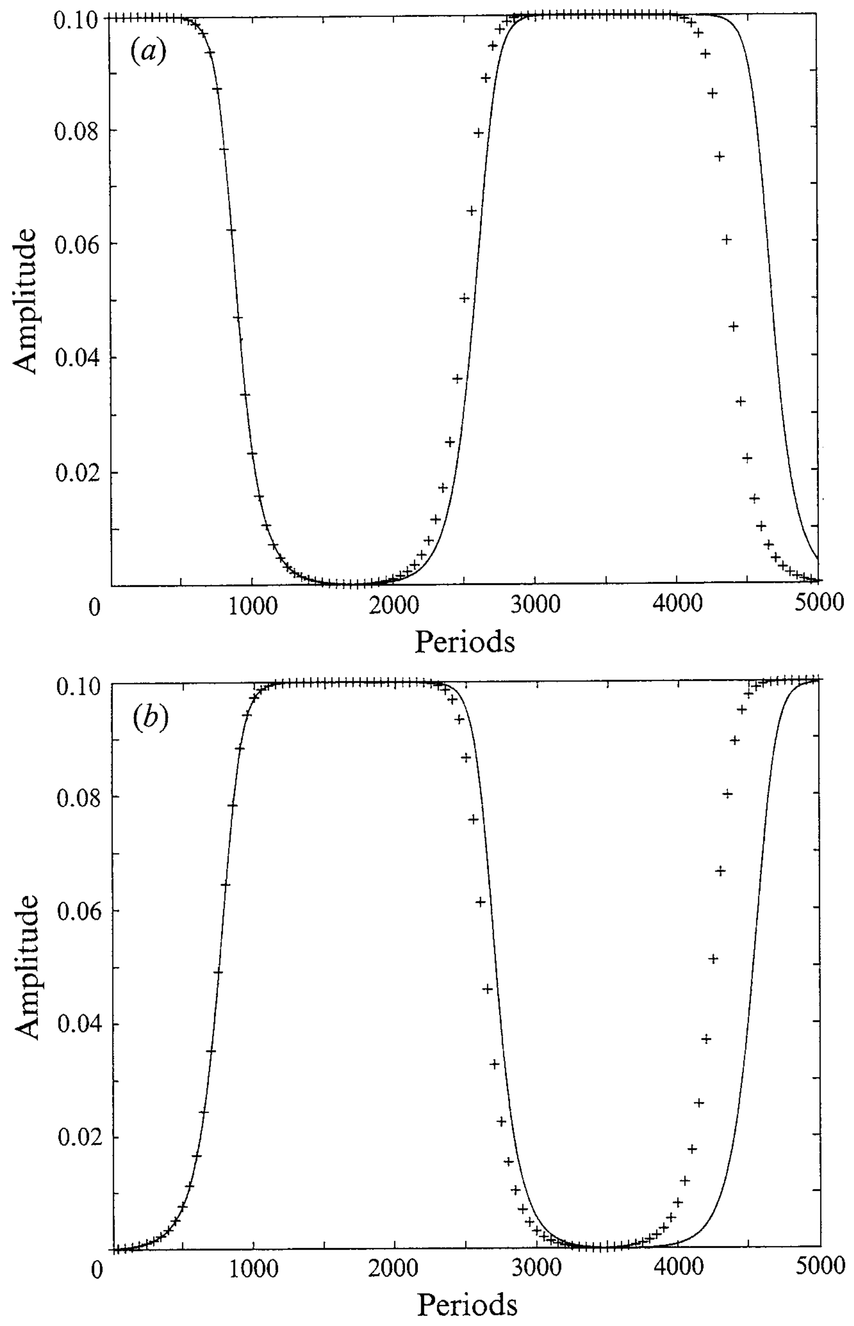

FIGURE 3. A comparison of the fully nonlinear solution for the amplitude of the first harmonic in (*a*) the x-direction and (*b*) the y-direction, drawn as solid curves, with the same amplitude calculated from the Zakharov model, shown as crosses. The initial condition consists of the two-dimensional standing wave of amplitude 0.1 in the x-direction disturbed by a standing wave of amplitude 0.0001 in the y-direction at a phase difference of $\pi/4$.

which with (4.7*a*) has the solution

$$\frac{r_1}{Q_1}=\frac{\exp(-C_0^{1/2}Q_1\,t)}{\delta+\exp(-C_0^{1/2}Q_1\,t)},\quad \frac{r_2}{Q_1}=\frac{\delta}{\delta+\exp(-C_0^{1/2}Q_1\,t)},$$

where $Q_1=0.1$, $\delta=10^{-8}$. The curves followed by the first 1800 wave periods of figure 3(*a*, *b*) are the square roots of these equations. The subsequent cyclic recurrence exhibited in figure 3(*a*, *b*) is described by the more accurate nonlinear representation in the solution (4.10) of equation (4.9).

5. Standing waves with resonating first and fourth harmonics

5.1. *Periodic standing waves*

The new standing waves with resonating first and fourth harmonics are dominated by the fundamental and fourth harmonics at wavenumbers $i, j, 4i, 4j$, with the remaining harmonics decreasing in magnitude in a Stokes ordering. The evolution equations (3.4*b, c*) and (3.5*b, c*) for such waves in the Zakharov model are

$$\mathrm{i}\frac{\mathrm{d}A_i}{\mathrm{d}t} = -\tfrac{1}{8}|A_i|^2A_i + \frac{-5+4\surd 2}{56}|A_j|^2A_i + \frac{-19+5\surd 17}{64}|A_{4j}|^2A_i - \tfrac{3}{16}A_i^*A_j^2, \quad (5.1a)$$

$$\mathrm{i}\frac{\mathrm{d}A_j}{\mathrm{d}t} = -\tfrac{1}{8}|A_j|^2A_j + \frac{-5+4\surd 2}{56}|A_i|^2A_j + \frac{-19+5\surd 17}{64}|A_{4i}|^2A_j - \tfrac{3}{16}A_j^*A_i^2, \quad (5.1b)$$

$$\mathrm{i}\frac{\mathrm{d}A_{4i}}{\mathrm{d}t} = -4|A_{4i}|^2A_{4i} + \frac{4(-5+4\surd 2)}{7}|A_{4j}|^2A_{4i} + \frac{-19+5\surd 17}{32}|A_j|^2A_{4i} - 6A_{4i}^*A_{4j}^2, \quad (5.1c)$$

and

$$\mathrm{i}\frac{\mathrm{d}A_{4j}}{\mathrm{d}t} = -4|A_{4j}|^2A_{4j} + \frac{4(-5+4\surd 2)}{7}|A_{4i}|^2A_{4j} + \frac{-19+5\surd 17}{32}|A_i|^2A_{4j} - 6A_{4j}^*A_{4i}^2. \quad (5.1d)$$

Solutions of these equations describing periodic waves are given by

$$A_i = a_i\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}\phi_i}, \quad A_j = a_j\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}\phi_j}, \quad A_{4i} = a_{4i}\mathrm{e}^{-2\mathrm{i}\Omega t+\mathrm{i}\phi_{4i}}, \quad A_{4j} = a_{4j}\mathrm{e}^{-2\mathrm{i}\Omega t+\mathrm{i}\phi_{4j}},$$

where $a_i, a_j, a_{4i}, a_{4j}, \phi_i, \phi_j, \phi_{4i}, \phi_{4j}, \Omega$ are real constants. Equations (5.1*a–d*) may be rewritten

$$\left(\Omega + \tfrac{1}{8}a_i^2 - \frac{-5+4\surd 2}{56}a_j^2 - \frac{-19+5\surd 17}{64}a_{4j}^2 + \tfrac{3}{16}a_j^2\,\mathrm{e}^{2\mathrm{i}(\phi_j-\phi_i)}\right)a_i = 0, \quad (5.2a)$$

$$\left(\Omega + \tfrac{1}{8}a_j^2 - \frac{-5+4\surd 2}{56}a_i^2 - \frac{-19+5\surd 17}{64}a_{4i}^2 + \tfrac{3}{16}a_i^2\,\mathrm{e}^{2\mathrm{i}(\phi_i-\phi_j)}\right)a_j = 0, \quad (5.2b)$$

$$\left(2\Omega + 4a_{4i}^2 - \frac{4(-5+4\surd 2)}{7}a_{4j}^2 - \frac{-19+5\surd 17}{32}a_j^2 + 6a_{4j}^2\,\mathrm{e}^{2\mathrm{i}(\phi_{4j}-\phi_{4i})}\right)a_{4i} = 0, \quad (5.2c)$$

and

$$\left(2\Omega + 4a_{4j}^2 - \frac{4(-5+4\surd 2)}{7}a_{4i}^2 - \frac{-19+5\surd 17}{32}a_i^2 + 6a_{4i}^2\,\mathrm{e}^{2\mathrm{i}(\phi_{4i}-\phi_{4j})}\right)a_{4j} = 0. \quad (5.2d)$$

Equations (5.2*a–d*) admit the two-dimensional standing wave solutions (I, equation (2.15))

$$a_j = a_{4j} = 0, \quad a_{4i}/a_i = \pm\tfrac{1}{4}, \quad \Omega = -\tfrac{1}{8}a_i^2, \quad (5.3a)$$

oscillating end to end in the x-direction, and

$$a_i = a_{4i} = 0, \quad a_{4j}/a_j = \pm\tfrac{1}{4}, \quad \Omega = -\tfrac{1}{8}a_j^2, \quad (5.3b)$$

oscillating end to end in the y-direction.

For the above two-dimensional solutions, Zakharov's equation does not provide information about the phases (see §2.4 in I for more details).

They also admit four distinct types of three-dimensional standing wave solutions. It will be shown that the first type, the fully three-dimensional standing waves, are linearly stable to three-dimensional disturbances. The other three types have at least one component of the first or fourth harmonics missing and are either shown or expected to be linearly unstable to disturbances containing the missing component.

The solutions for the first type of three-dimensional standing wave, in which the first and fourth harmonics have both x- and y-components present, are

$$\left.\begin{aligned} a_j^2 = a_i^2, \quad a_{4j}^2 = a_{4i}^2, \quad \frac{a_{4i}^2}{a_i^2} &= \frac{(3-\surd 2)/7+(-19+5\surd 17)/32 \pm \frac{3}{8}}{16(3-\surd 2)/7+(-19+5\surd 17)/32 \pm 6}, \\ \Omega &= -\left(\frac{3-\surd 2}{14} \pm \frac{3}{16}\right) a_i^2 + \frac{-19+5\surd 17}{64} a_{4i}^2, \end{aligned}\right\} \qquad (5.4a)$$

where the $\pm$ sign in the equation for Ω is in the same order as the numerator of the previous equation. There are four combinations of the signs in (5.4a) resulting from four combinations of the phase differences, but only two of these have solutions in which the amplitudes are real, namely

(i) $\phi_j = \phi_i, \ \phi_{4j} = \phi_{4i}$

$$a_j^2 = a_i^2, \quad a_{4j}^2 = a_{4i}^2, \quad \frac{a_{4i}^2}{a_i^2} = 0.067392, \quad \frac{a_{4i}}{a_i} = 0.259600, \quad \Omega = -0.299069 a_i^2, \qquad (5.4b)$$

(ii) $\phi_j = \phi_i \pm \frac{1}{2}\pi, \quad \phi_{4j} = \phi_{4i} \pm \frac{1}{2}\pi$

$$a_j^2 = a_i^2, \quad a_{4j}^2 = a_{4i}^2, \quad \frac{a_{4i}^2}{a_i^2} = 0.042142, \quad \frac{a_{4i}}{a_i} = 0.205285, \quad \Omega = 0.075293 a_i^2. \qquad (5.4c)$$

Solutions of the fully nonlinear problem have been found which correspond to the first Zakharov solution (5.4b) and possibly to the second solution (5.4c). Equations (5.2a–d) place no constraint on the phase differences between the first and fourth harmonics in the above Zakharov solutions.

Fourier series expansions (2.2b, c) for the above three-dimensional standing waves were sought by the fixed point method of I, §3.2, to satisfy the fully nonlinear boundary conditions (2.1b, c), starting from the Zakharov solutions (5.4b, c) as first approximations. The simplest waves of this type are those in (i) above for which both components of the first and fourth harmonics have the same phase. Their frequencies, expanded as polynomials in the fundamental amplitude a_{101} over the range $0 < \epsilon < 0.1$ (using the NAG subroutine E02ADF), are found to have the leading terms

$$\omega = 1.0000000 - 0.299068 a_{101}^2 + \ldots, \qquad (5.4d)$$

and their fourth harmonics have the leading term

$$a_{402} = 0.259604 a_{101} + \ldots. \qquad (5.4e)$$

The excellent agreement between (5.4b) and (5.4d, e) gives confidence in the results because the two derivations are independent. (Equivalent standing wave solutions in which the amplitude ratio in (5.4b) is negative, corresponding to a phase difference of π, have been found to have the same excellent agreement with the Zakharov solutions.)

Three fully nonlinear families of standing waves have been found which correspond partially to those described in (ii) above. The phases in the fully nonlinear standing waves are, in the first family,

$$\phi_i = 0, \quad \phi_j = \tfrac{1}{2}\pi, \quad \phi_{4i} = \tfrac{1}{4}\pi, \quad \phi_{4j} = \tfrac{3}{4}\pi,$$

in the second family,

$$\phi_i = 0, \quad \phi_j = \tfrac{1}{2}\pi, \quad \phi_{4i} = \tfrac{3}{4}\pi, \quad \phi_{4j} = \tfrac{1}{4}\pi,$$

and in the third family,

$$\phi_i = 0, \quad \phi_j = \tfrac{1}{2}\pi, \quad \phi_{4i} = -\tfrac{1}{4}\pi, \quad \phi_{4j} = -\tfrac{3}{4}\pi.$$

(The phase ϕ_i is set equal to zero and the phase difference $\phi_j - \phi_i$ is chosen to be positive in each family.) It is noted that the phase differences between the x- and y-components of the first and fourth harmonics are all $\frac{1}{2}\pi$, in agreement with (5.4c). The frequency of the first of these standing waves, expanded as a polynomial in the fundamental amplitude a_{101} over a range near $\epsilon = 0.05$ (with the phase chosen as above so that b_{101} is zero), is found to have the leading terms

$$\omega = 1.00000 + 0.0718 a_{101}^2 + \dots, \tag{5.4f}$$

in reasonable agreement with (5.4c). The x-component of its fourth harmonic has the leading term

$$(a_{402}^2 + b_{402}^2)^{1/2} = 0.112 a_{101} + \dots, \tag{5.4g}$$

which does not agree with (5.4c). It is not known why the fully nonlinear standing wave solutions for these three families are not completely consistent with the Zakharov solutions.

At the smallest values of ϵ, the fixed point method converges strongly towards solutions for the amplitudes of three-dimensional standing waves of this type, but fails to converge towards reproducible solutions for the corresponding phases. This is not a problem in the standing waves described by (5.4a, b) because the phase differences there are all fixed at 0 or π. It indicates that the phases of the fourth-harmonic components relative to the first-harmonic components are determined at a higher order of interaction than the corresponding amplitudes.

In the second type of three-dimensional standing waves, one component of each of the first and fourth harmonics is missing. Either the first harmonic is in the x-direction and the fourth harmonic is in the y-direction, or vice versa. Their solutions are

$$\left.\begin{aligned} a_j = a_{4i} = 0, \quad \frac{a_{4j}^2}{a_i^2} = \frac{-11+5\sqrt{17}}{109+5\sqrt{17}}, \quad \Omega = -\frac{119+95\sqrt{17}}{32(109+5\sqrt{17})} a_i^2, \\ \frac{a_{4j}}{a_i} = \pm 0.272369, \quad \Omega = -0.123127 a_i^2, \end{aligned}\right\} \tag{5.5a}$$

or the same with $\boldsymbol{i}$ and $\boldsymbol{j}$ interchanged. Equations (5.2a–d) place no constraint on the phase difference between the fourth harmonic and the first harmonic in the above standing wave solutions. The only solutions of this type which have been found by the fully nonlinear calculations have phase differences 0 and π between the fourth and first harmonics, corresponding to the $\pm$ signs respectively in (5.5a).

The Fourier series expansions (2.2b, c) for these three-dimensional standing waves are calculated by the fixed point method of I, §3.2, to satisfy the fully nonlinear boundary conditions (2.1b, c), using the Zakharov solution (5.5a) as a first approximation. The frequency ω for the three-dimensional standing waves, expanded as polynomials in the fundamental amplitude a_{101} over the range $0 < \epsilon < 0.1$ is found to have the leading terms

$$\omega = 1.0000000 - 0.123130 a_{101}^2 + \dots, \tag{5.5b}$$

and the fourth harmonic has the leading term

$$a_{042} = 0.272369 a_{101} + \dots. \tag{5.5c}$$

The excellent agreement between (5.5*a*) and (5.5*b*, *c*) gives confidence in the results because the two derivations are independent.

In the third type of three-dimensional standing waves, one component of the first harmonic is missing. The first harmonic is either in the *x*-direction or in the *y*-direction, and the fourth harmonic has both components. The Zakharov solutions with real amplitudes are

$$a_j = 0, \quad a_{4i}^2/a_i^2 = 0.044\,067, \quad a_{4j}^2/a_i^2 = 0.012992, \quad \Omega = -0.124672a_i^2, \tag{5.6}$$

or the same with i and j interchanged. The fully nonlinear calculations, using the Zakharov solution (5.6) as a first approximation, converged to other fully nonlinear solutions in the neighbourhood of (5.6), particularly that corresponding to (5.3*a*), but failed to converge to a fully nonlinear solution that could be identified with (5.6).

In the fourth type of three-dimensional standing waves, one component of the fourth harmonic is missing. The first harmonic has both components and the fourth harmonic is either in the *x*-direction or in the *y*-direction. Their solutions are

$$a_{4j} = 0, \quad a_j^2/a_i^2 = 0.922812, \quad \frac{a_{4i}^2}{a_i^2} = 0.155249, \quad \Omega = -0.287203a_i^2, \tag{5.7}$$

or the same with i and j interchanged. The fully nonlinear calculations, using the Zakharov solution (5.7) as a first approximation, converged to other fully nonlinear solutions in the neighbourhood of (5.7), particularly that corresponding to (4.3*c*), but failed to converge to a fully nonlinear solution that could be identified with (5.7).

5.2. *Non-periodic solutions of the evolution equations*

When the complex amplitudes in (5.1*a*–*d*) are replaced by the real variables

$$\left.\begin{aligned} r_1 = |A_i|^2, \quad r_2 = |A_j|^2, \quad r_3 = A_i A_j^* + A_i^* A_j, \quad r_4 = (A_i A_j^* - A_i^* A_j)/i, \\ r_5 = |A_{4i}|^2, \quad r_6 = |A_{4j}|^2, \quad r_7 = A_{4i} A_{4j}^* + A_{4i}^* A_{4j}, \quad r_8 = (A_{4i} A_{4j}^* - A_{4i}^* A_{4j})/i, \end{aligned}\right\} \tag{5.8a}$$

from which

$$r_3^2 + r_4^2 = 4r_1 r_2, \quad r_7^2 + r_8^2 = 4r_5 r_6, \tag{5.8b}$$

it may be shown that

$$r_{1t} = \tfrac{3}{16} r_3 r_4, \quad r_{2t} = -\tfrac{3}{16} r_3 r_4, \tag{5.9a, b}$$

$$r_{3t} = -\frac{25+8\sqrt{2}}{112}(r_1 - r_2)\, r_4 - \frac{-19+5\sqrt{17}}{64}(r_5 - r_6)\, r_4, \tag{5.9c}$$

$$r_{4t} = -\frac{17-8\sqrt{2}}{112}(r_1 - r_2)\, r_3 + \frac{-19+5\sqrt{17}}{64}(r_5 - r_6)\, r_3, \tag{5.9d}$$

$$r_{5t} = 6r_7 r_8, \quad r_{6t} = -6r_7 r_8, \tag{5.9e, f}$$

$$r_{7t} = -\frac{2(25+8\sqrt{2})}{7}(r_5 - r_6)\, r_8 - \frac{-19+5\sqrt{17}}{32}(r_1 - r_2)\, r_8, \tag{5.9g}$$

$$r_{8t} = -\frac{2(17-8\sqrt{2})}{7}(r_5 - r_6)\, r_7 + \frac{-19+5\sqrt{17}}{32}(r_1 - r_2)\, r_7, \tag{5.9h}$$

where the t-subscript denotes the derivative.

It follows from (5.9*a*, *b*) that

$$r_1 + r_2 = Q_1, \tag{5.10a}$$

and from (5.9*e*,*f*) that

$$r_5 + r_6 = Q_2, \tag{5.10b}$$

where Q_1, Q_2 are constants. No analytical solutions of the full set of equations (5.9) have been found, but numerical solutions (using the NAG variable-order variable step backward differentiation integrator D02EBF) may be calculated without difficulty.

The evolution equations can be solved analytically when the first harmonic remains in the x-direction, or in other words,

$$A_j(t) = 0$$

for all t, satisfying (5.1 b) and its derivatives. This means that

$$r_2 = r_3 = r_4 = 0$$

in (5.8 a) and

$$r_1 = Q_1$$

in (5.10 a), showing that the first harmonic has a constant amplitude. Equations (5.9 g, h), with substitutions from (5.9 e, f) and (5.10 b), integrate to

$$r_7^2 = \frac{2(25+8\sqrt{2})}{21} r_5(Q_2 - r_5) - \frac{-19+5\sqrt{17}}{96} Q_1 r_5 + Q_3, \tag{5.11a}$$

$$r_8^2 = \frac{2(17-8\sqrt{2})}{21} r_5(Q_2 - r_5) + \frac{-19+5\sqrt{17}}{96} Q_1 r_5 - Q_3, \tag{5.11b}$$

in which Q_3 is a constant of integration. Equation (5.9 e) can then be rewritten

$$r_{5t}^2 = \frac{16(25+8\sqrt{2})(17-8\sqrt{2})}{49}\left(-r_5^2 + \left(Q_2 - \frac{7(-19+5\sqrt{17})}{64(25+8\sqrt{2})} Q_1\right) r_5 + \frac{21 Q_3}{2(25+8\sqrt{2})}\right)$$
$$\times\left(-r_5^2 + \left(Q_2 + \frac{7(-19+5\sqrt{17})}{64(17-8\sqrt{2})} Q_1\right) r_5 - \frac{21 Q_3}{2(17-8\sqrt{2})}\right), \tag{5.11c}$$

where Q_1, Q_2, Q_3 are constants determined by the initial conditions, through (5.10 a), (5.10 b) and (5.11 a).

The phase-plane diagram for (5.11 c) is of similar form to figure 1 except that the expressions containing Q_1 cause a loss of symmetry of the quartic curves about the centre B. Equation (5.11 c), like (4.9), has solutions in terms of elliptic functions, but unlike (4.9) the solutions are not expressible in simple form because of the lack of symmetry of the quartic curves. The symmetry-breaking expressions containing Q_1 are an essential part of the equation because they originate from the resonant interaction between the fourth harmonic and the underlying first harmonic.

A solution of (5.11 c) is developed for a particular example, that of the evolution of the SA two-dimensional standing wave of I with $\epsilon = 0.1$ when a transverse standing wave with wavenumber 4 of amplitude 0.01 is superposed on it. The initial phase difference between the waves is set at $\pi/4$. The dominant harmonics of the SA wave are the first harmonic of amplitude 0.097043 and the fourth harmonic of amplitude 0.023452, both being in the x-direction with the same phase, which is set equal to zero. In the notation of (5.1 a–d), the initial conditions are

$$A_i = 0.097043, \quad A_j = 0, \quad A_{4i} = 0.023452, \quad A_{4j} = 0.01\,\mathrm{e}^{\mathrm{i}\pi/4}, \tag{5.12a}$$

which in (5.8 a) become

$$r_5 = 0.023452^2, \quad r_6 = 0.0001, \quad r_7 = -r_8 = 0.00023452\sqrt{2}. \tag{5.12b}$$

The constants in (5.11 c) are therefore

$$Q_1 = 0.009417, \quad Q_2 = 0.000650, \quad Q_3 = 6.95\times 10^{-9}, \tag{5.12c}$$

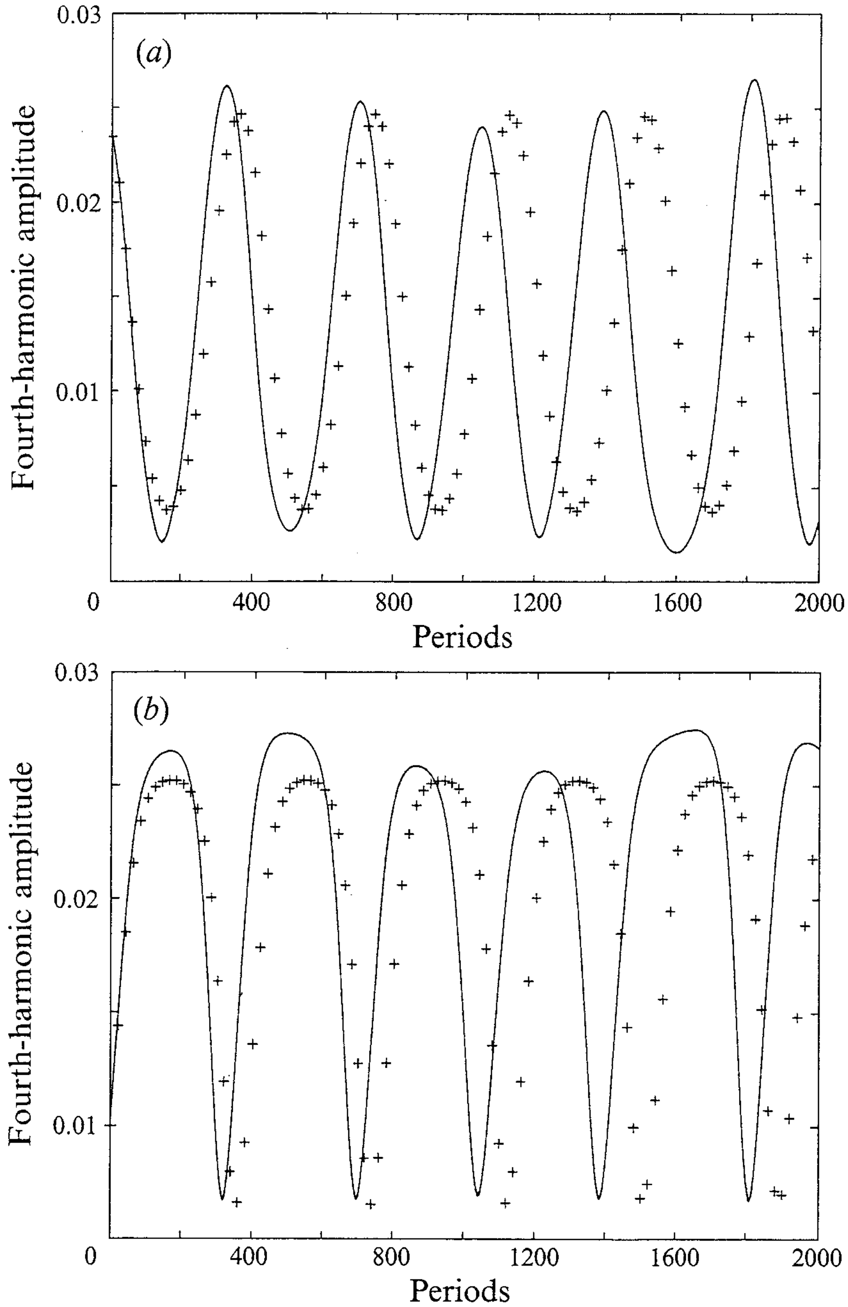

FIGURE 4. A comparison of the fully nonlinear solution for the amplitude of the fourth harmonic in (*a*) the x-direction and (*b*) the y-direction, drawn as solid curves, with the same amplitude calculated from the Zakharov model, shown as crosses. The initial condition consists of the two-dimensional standing wave SA with $\epsilon = 0.1$ in the x-direction at a phase difference of $\pi/4$ with a fourth-harmonic standing wave of amplitude 0.01 in the y-direction.

and the roots r_5 of the quartic polynomial in (5.11 *c*) are

$$\left.\begin{aligned} \alpha_1 &= -3\times 10^{-6}, \quad \alpha_2 = 1.4\times 10^{-5} = 0.0037^2, \\ \alpha_3 &= 0.000\,607 = 0.0246^2, \quad \alpha_4 = 0.000929. \end{aligned}\right\} \qquad (5.12\,d)$$

The first and third roots arise from (5.11 *a*) and the second and fourth roots from (5.11 *b*). The solution of (5.11 *c*) lies between the roots α_2, α_3 where the quartic polynomial is positive. The solution for $|A_{4i}| = r_5^{1/2}$ is drawn as crosses at intervals of 20 periods in figure 4(*a*), and the solution for $|A_{4j}| = r_6^{1/2}$ as crosses at intervals of 20 periods in figure 4(*b*).

The fully nonlinear time evolution of this example is calculated as a check on the validity of the above results. The expansions (2.5 a, b) are truncated so that they contain all wavenumbers (l, m) such that $l+m \leqslant 9$ with the exception of $(0,0)$. This gives a system of 108 amplitudes a_{lm}, c_{lm}, whose evolution is calculated (I, §3.3) with a local error tolerance of 10^{-11}. The fully nonlinear solution for $(a_{402}^2+b_{402}^2)^{1/2}$, which is the same as $r_5^{1/2}$ in the Zakharov model, is compared with it in figure 4(a), where the solid curve is the fully nonlinear solution. A similar comparison is made between the fully nonlinear solution for $(a_{042}^2+b_{042}^2)^{1/2}$ and $r_6^{1/2}$ in figure 4(b).

Although the initial agreement in both figures is excellent, the fully nonlinear solution diverges from the Zakharov solution after about 100 wave periods. The two solutions have the same form at later times but they differ in the locations of the amplitude maxima and minima. The difference appears to be due to approximations in the Zakharov model. Although the deduction from (5.1 b) that there are standing wave solutions for which $A_j(t) = 0$ for all t is confirmed by the fully nonlinear calculations, the associated deduction from (5.10 a) that $|A_i(t)|$ remains constant is not confirmed by the fully nonlinear calculations. The quantity Q_1 is a measure of the energy of the first harmonic, and instead of remaining constant it is found to vary in slow time by about 1 % in the fully nonlinear calculation. The quantity Q_2 is a measure of the total energy of the two components of the fourth harmonic, and in order that the total energy of the system remains constant it is found to vary in slow time by about 15 % in the fully nonlinear calculation. With Q_1 and Q_2 constant in a local sense only, it is not surprising that the Zakharov solutions of (5.11 c) agree only locally in form with the fully nonlinear solutions. The variation over long times of Q_1 and Q_2 is probably due to higher-order resonant interactions neglected in the Zakharov model.

5.3. *Stability of the three-dimensional periodic standing waves*

There are four distinct types of new three-dimensional periodic standing waves with resonating first and fourth harmonics, given by (5.4 a–c), (5.5 a), (5.6) and (5.7). Only the first type is fully three-dimensional with both the x- and y-components of the first and fourth harmonics present. The other three types have at least one of these components missing.

In the first type, both components of the first and fourth harmonics are not only present but the amplitudes in each of the pairs of components are equal (5.4 a). It was shown (§4.3) for the Stokes type of standing waves that when both components of the first harmonic are present and equal in a periodic standing wave, the wave is stable to disturbances containing these components. This property is also true here, and was tested by calculating the fully nonlinear time evolution of perturbed examples of this first type at small to moderate root-mean energies $\epsilon = 0.05$, 0.1, 0.15. Each was integrated over 2000 wave periods, using the same program as that for figure 4(a, b), and none showed any instability. The typical time evolution consists of a slow time oscillation in antiphase of the two components of the first harmonic, and a slow time oscillation in antiphase, at a different frequency from the first harmonic, of the two components of the fourth harmonic.

In the second type, only the first harmonic is in the x-direction and the fourth harmonic is in the y-direction, or vice versa. It was shown (§4.4) for the Stokes type of standing waves that when one component in three dimensions of the first harmonic is missing from a periodic standing wave, the wave is linearly unstable to disturbances containing the missing component, although it returns cyclically near to the initial conditions. It can be expected therefore that the second type of new three-dimensional periodic standing waves (5.5 a) is linearly unstable to three-dimensional disturbances containing the missing components of the first or the fourth harmonics, with the possibility of cyclic recurrence.

These properties were tested with a particular example, that of the evolution of the new three-dimensional wave of the second type (5.5*a*) with $\epsilon = 0.1$ when the only disturbance imposed on it is the numerical error in its calculation. The dominant harmonics of the wave are the x-component of the first harmonic with an amplitude 0.0965 and the y-component of the fourth harmonic with an amplitude 0.0261, both with the same phase, which is set equal to zero. The fourth harmonic has a small x-component due to the quartic interaction of the first harmonic with itself. In the notation of (5.1*a–d*), the initial conditions are

$$A_i = 0.0965, \quad A_j = 0.0, \quad A_{4i} = -0.0001, \quad A_{4j} = 0.0261, \tag{5.13}$$

which in (5.8*a*) become initial conditions for $r_1, \ldots, r_8$. The fully nonlinear time evolution of this example is similar to that illustrated in figure 4(*a, b*) reversed, that is, the x-component of the fourth harmonic follows an evolution similar to figure 4(*b*) and the y-component of the fourth harmonic similar to figure 4(*a*). The y-component of the first harmonic remains zero for all time in the fully nonlinear time evolution, which means that the Zakharov solution is given by (5.11*c*). It is found to diverge from the fully nonlinear solution to the same extent as is illustrated in figure 4(*a, b*) and for the same probable reasons. These calculations show that the new three-dimensional wave of the second type (5.5*a*) with $\epsilon = 0.1$ is linearly unstable, with the instability in the present example arising only in the fourth harmonic. It does not occur in the first harmonic in this example because the y-component of the first harmonic is exactly zero initially, and the instability is never seeded. Cyclic recurrence occurs in this example, with the time evolution returning close to the initial conditions.

By allowing the disturbance to arise from numerical error in the example above, no opportunity was given for instability to occur in the first harmonic. For this reason, a more general example was tested, that of the evolution of the new three-dimensional wave of the second type (5.5*a*) with $\epsilon = 0.1$ when it is seeded with a periodic disturbance in the y-component of the first harmonic of amplitude 0.0001. The initial phase difference between the two components is set at $\pi/4$. In the notation of (5.1*a–d*), the initial conditions are

$$A_i = 0.0965, \quad A_j = 0.0001\,\mathrm{e}^{i\pi/4}, \quad A_{4i} = -0.0001, \quad A_{4j} = 0.0261, \tag{5.14a}$$

which in (5.8*a*) become initial conditions for $r_1, \ldots, r_8$. The set of equations (5.9) is solved numerically, and the solutions for

$$|A_i| = r_1^{1/2}, \quad |A_j| = r_2^{1/2}, \quad |A_{4i}| = r_5^{1/2}, \quad |A_{4j}| = r_6^{1/6} \tag{5.14b}$$

are drawn as crosses at intervals of 20 periods in figure 5(*a–d*) respectively.

The fully nonlinear time evolution of this example was calculated as a check on the validity of the above results, using the same program as that described for figure 4(*a, b*). The fully nonlinear solutions for

$$(a_{101}^2 + b_{101}^2)^{1/2}, \quad (a_{011}^2 + b_{011}^2)^{1/2}, \quad (a_{402}^2 + b_{402}^2)^{1/2}, \quad (a_{042}^2 + b_{042}^2)^{1/2}$$

are drawn as solid curves in figure 5(*a–d*) respectively.

Although the initial agreement in the figures is excellent, the fully nonlinear solution diverges from the Zakharov solution in the first harmonic after about 1100 wave periods and in the fourth harmonic after about 100 wave periods. The divergence appears to be due to approximations in the Zakharov model. If figures 2(*a, b*) and 3(*a, b*) are compared with figure 4(*a, b*), it can be seen that the agreement between the fully nonlinear evolution and the Zakharov evolution persists over much longer times

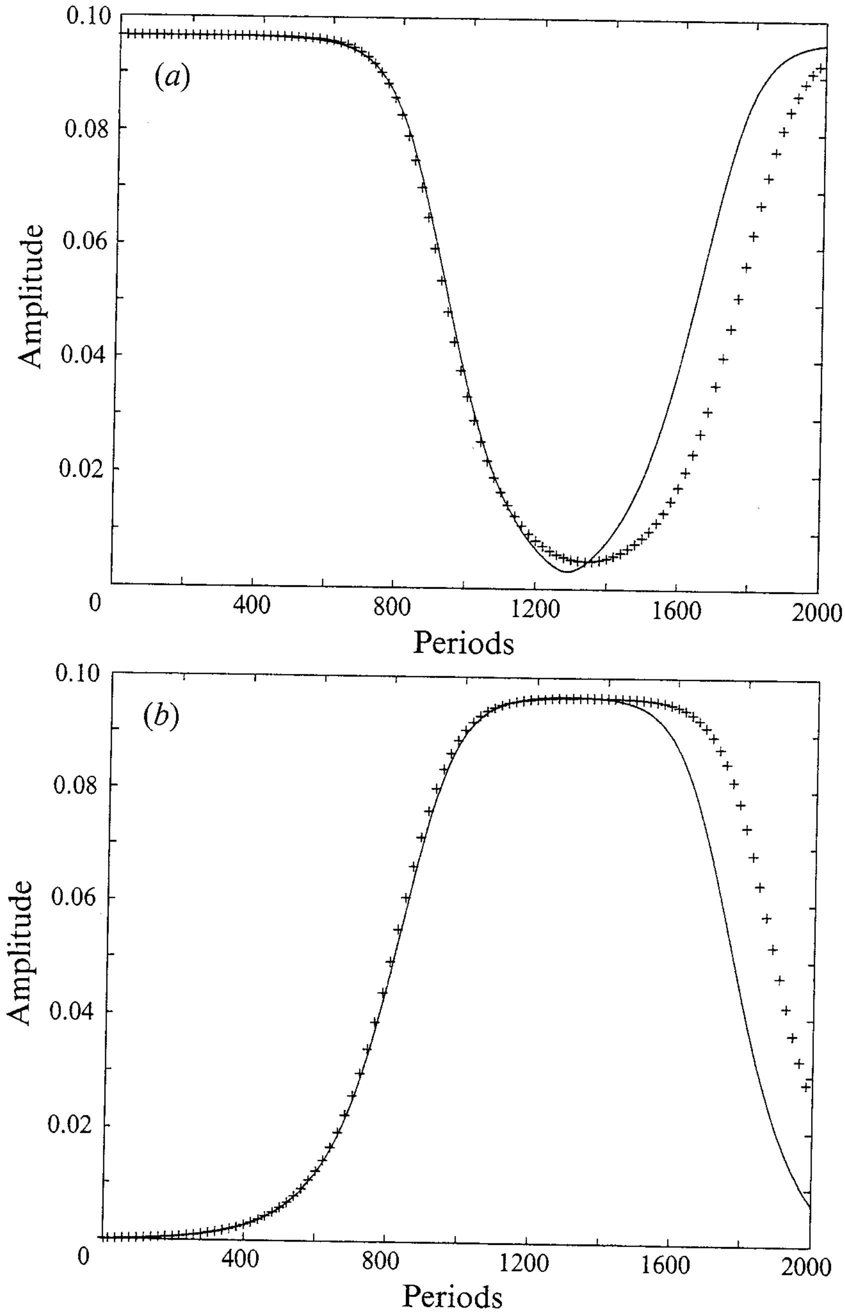

FIGURE 5(a,b). For caption see facing page.

when it occurs predominantly in the first harmonic rather than in the fourth harmonic. It is interesting that this property is still true in figure 5(a–d) when the evolution occurs simultaneously in both the first and fourth harmonics, which indicates that the nonlinear interactions between the first and fourth harmonics are much weaker than the nonlinear interactions between the components of either of the harmonics. Figure 5(a–d) confirms that the new three-dimensional wave of the second type (5.5a) with $\epsilon = 0.1$ is linearly unstable to disturbances containing any of the missing components of the first and fourth harmonics.

The new three-dimensional wave of the second type (5.5a) with $\epsilon = 0.05$ is also linearly unstable, and exhibits cyclic recurrence over a much longer time than that in figures 4 or 5, while the wave of this type with $\epsilon = 0.15$ exhibits cyclic recurrence over a much shorter time than that in figures 4 or 5. Solutions for the new three-dimensional wave of the third type (5.6) and the fourth type (5.7) failed to converge and cannot

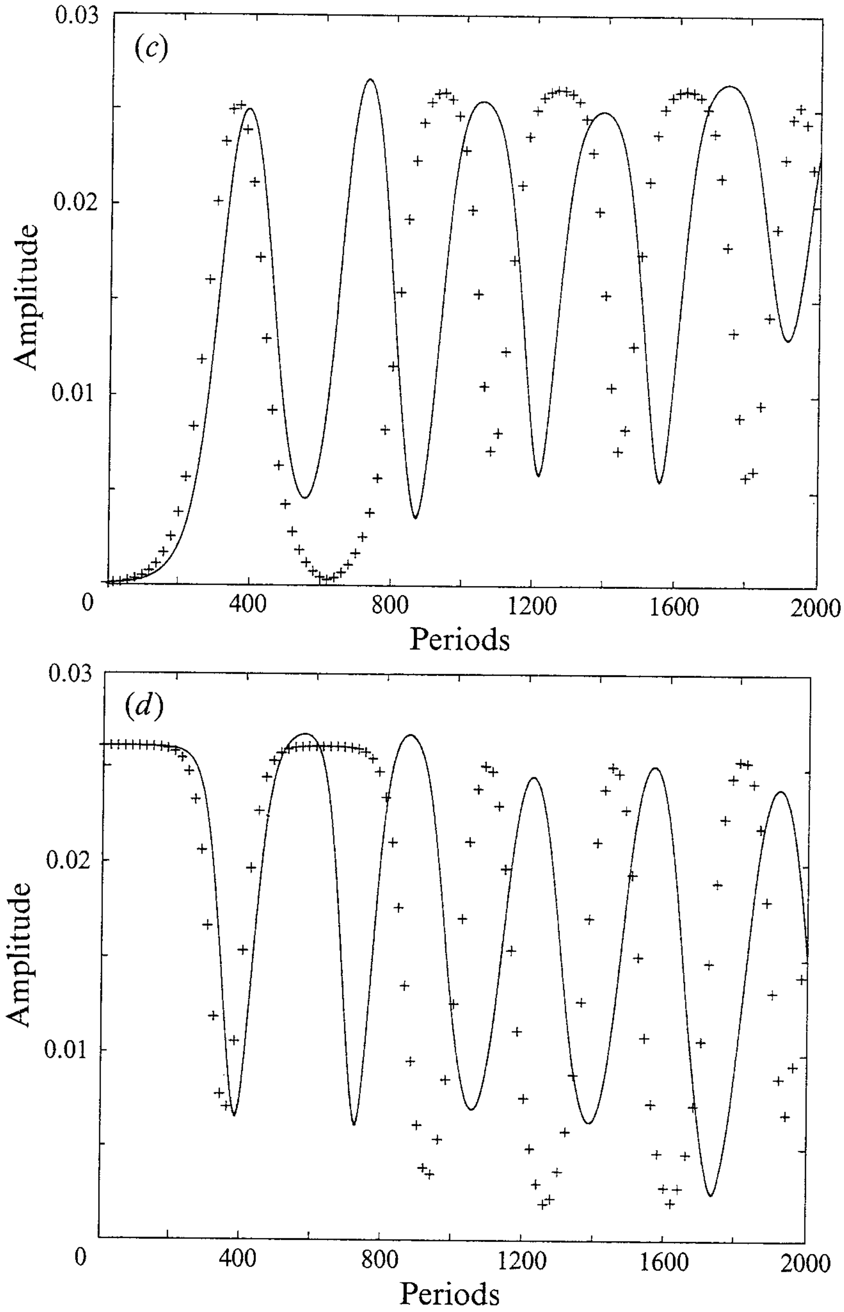

FIGURE 5. (*a*, *b*) A comparison of the fully nonlinear solution for the amplitude of the first harmonic in (*a*) the x-direction and (*b*) the y-direction, drawn as solid curves, with the same amplitude calculated from the Zakharov model, shown as crosses. (*c*, *d*) A comparison of the fully nonlinear solution for the amplitude of the fourth harmonic in (*c*) the x-direction and (*d*) the y-direction, drawn as solid curves, with the same amplitude calculated from the Zakharov model, shown as crosses. In all cases, the initial condition consists of the new three-dimensional standing wave at $\epsilon = 0.1$ with the first harmonic in the x-direction at a phase difference of $\pi/4$ with a first harmonic of amplitude 0.0001 in the y-direction.

therefore be tested for stability. The failure to converge, and the behaviour of the second type above, together suggest that these waves are unstable also.

5.4. *Three-dimensional instability of the two-dimensional standing waves*

The two families of two-dimensional periodic standing waves with resonating first and fourth harmonics are described by (5.3 *a*) and (5.3 *b*). A wave solution in the first family satisfies (5.11 *c*) with $r_{5t} = 0$, with a value of r_5 which lies at a point in the phase-plane diagram equivalent to D on the highest curve in figure 1. If a small three-dimensional perturbation is applied to the fourth harmonic of such a wave, so that $r_6 > 0$, the motion is governed by (5.11 *c*) with $r_{5t} \neq 0$. The curve describing the right-hand side of (5.11 *c*) is then obtained approximately by slightly lowering the highest curve in the phase-plane diagram similar to figure 1. For the reasons described in §4.4, the evolution of the perturbed standing wave (5.3 *a*) is described by a point which moves along the curve with r_5 decreasing, $r_{5t} < 0$, $r_{5t}^2 > 0$, until it reaches the zero of r_{5t} near O. This behaviour is linearly unstable with r_5 returning cyclically to the neighbourhood of its initial point.

The fully nonlinear time evolution of a particular example was calculated and compared with the results above, as a check on their validity. The example chosen was a two-dimensional standing wave in the x-direction, (4.3 *a*), with $\epsilon = 0.1$, which is disturbed at $t = 0$ by a two-dimensional standing wave in the y-direction, (4.3 *b*), with $\epsilon = 0.0001$, at an initial phase difference of $\pi/4$. The calculation was made with the same number of harmonics and to the same precision as that described in §5.2. The initial behaviour of the two components of the fourth harmonic is similar to that illustrated in figure 3(*a*, *b*), with the x-component remaining almost constant for the first 200 wave periods while the y-component grows exponentially. The two components then enter into an approximate form of cyclic recurrence, with divergence between the fully nonlinear and Zakharov solutions similar to that shown in figure 4(*a*, *b*).

6. Alternative formulation for three-dimensional standing waves

The three-dimensional standing waves described in §§2–5 consist of two-dimensional free wave components in the x- and y-directions (such as (2.2 *a*)) together with their three-dimensional bound wave components. The three-dimensional standing waves developed in previous investigations (see (1.1)) have free components in a square basin such as

$$a_{111} \cos x \cos y \cos 2^{1/4} t, \quad a_{442} \cos 4x \cos 4y \cos 2^{5/4} t \qquad (6.1\,a)$$

together with their bound components. The two formulations are compared here. It is noted that

$$a_{111} \cos x \cos y \cos 2^{1/4} t = \frac{a_{111}}{2} \cos (x+y) \cos 2^{1/4} t + \frac{a_{111}}{2} \cos (x-y) \cos 2^{1/4} t, \qquad (6.1\,b)$$

which means that the three-dimensional standing wave component of amplitude a_{111} and unit wavenumber in both the x- and y-directions is the sum of two two-dimensional waves each of amplitude $a_{111}/2$ and wavenumber $\surd 2$ in two orthogonal directions. The three-dimensional wave oscillates in the square basin $0 < x < \pi$, $0 < y < \pi$, while the two two-dimensional waves are placed more naturally in the square basin $0 < x-y < \pi$, $0 < x+y < \pi$, with slope parameters $1/\surd 2$ that of the single wave. Equation (6.1 *b*) shows that the standing waves (4.3 *c*) and (5.4 *b*) may be reformulated in terms of three-dimensional free wave components such as (6.1 *a*) because the phases of the two two-dimensional standing waves in these solutions are the same, but that all other three-dimensional standing wave solutions in §§4 and 5 need the formulation used in those sections.

The formulation based on two-dimensional free wave components such as (2.2 *a*) is appropriate for a square basin because the frequencies of two-dimensional standing waves parallel to the two pairs of walls are the same. It would also be appropriate for rectangular basins in which the ratios of the lengths of the sides are 4, 9, ..., because the ratios of the frequencies of two-dimensional standing waves parallel to the two pairs of walls are then integers. The fully nonlinear and Zakharov theories using the free wave components (6.1 *a*) are described now for a square basin to confirm the Zakharov theory in §3, and because this approach is necessary for general rectangular basins.

The Fourier series expansions (before truncation) based on the free wave components (6.1 *a*) are

$$\eta = \sum_{l=1}^{\infty} \sum_{m=l \bmod 2}^{\infty} \sum_{n=l \bmod 2}^{\infty} \cos lx \cos my (a_{lmn} \cos n\omega t + b_{lmn} \sin n\omega t), \tag{6.2a}$$

and

$$\phi = \sum_{l=0}^{\infty} \sum_{m=l \bmod 2}^{\infty} \sum_{n=l \bmod 2}^{\infty} \cos lx \cos my\, e^{(l^2+m^2)^{1/2} z} (c_{lmn} \cos n\omega t + d_{lmn} \sin n\omega t), \tag{6.2b}$$

where l, m, n are either all even or all odd, the coefficients a_{lmn}, b_{lmn}, c_{lmn}, d_{lmn} are constants, and $\omega(\sim 2^{1/4})$ is the nonlinear frequency of the fundamental harmonic.

Zakharov's equation (3.1) for standing waves with free components having non-dimensional wavenumbers

$$\pm \boldsymbol{i} \pm \boldsymbol{j}, \quad \pm 4\boldsymbol{i} \pm 4\boldsymbol{j},$$

has the wave component

$$B(\boldsymbol{k}, t) = B_{11}(t)\,[\delta(\boldsymbol{k}-\boldsymbol{i}-\boldsymbol{j}) + \delta(\boldsymbol{k}+\boldsymbol{i}-\boldsymbol{j}) + \delta(\boldsymbol{k}-\boldsymbol{i}+\boldsymbol{j}) + \delta(\boldsymbol{k}+\boldsymbol{i}+\boldsymbol{j})]$$
$$+ B_{44}(t)\,[\delta(\boldsymbol{k}-4\boldsymbol{i}-4\boldsymbol{j}) + \delta(\boldsymbol{k}+4\boldsymbol{i}-4\boldsymbol{j}) + \delta(\boldsymbol{k}-4\boldsymbol{i}+4\boldsymbol{j}) + \delta(\boldsymbol{k}+4\boldsymbol{i}+4\boldsymbol{j})], \tag{6.3a}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ are the unit vectors in the x, y-directions. The dependent variables $B_{11}(t)$, $B_{44}(t)$ are replaced by $A_{11}(t)$, $A_{44}(t)$ where

$$\left.\begin{aligned} B_{11}^2 &= \frac{\pi^2}{8 \times 2^{1/4}} A_{11}^2, \quad A_{11} = a_{i+j} + \mathrm{i} 2^{1/4} b_{i+j}, \\ B_{44}^2 &= \frac{\pi^2}{8 \times 2^{5/4}} A_{44}^2, \quad A_{44} = a_{4i+4j} + \mathrm{i} 2^{5/4} b_{4i+4j}, \end{aligned}\right\} \tag{6.3b}$$

$a_{\boldsymbol{m}}$ is the complex Fourier amplitude of the wave component with wavenumber $\boldsymbol{m}$, and $b_{\boldsymbol{m}}$ is the corresponding complex amplitude of the velocity potential on the free surface. (Compare (3.3) or I, equation (2.14 *b*).) Substitution of (6.3 *a*) into Zakharov's equation (3.1) (with the superscript (2) omitted), yields

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}B_{11}}{\mathrm{d}t} = {} & [T_{i+j,\, i+j,\, i+j,\, i+j} + 2T_{i+j,\, -i-j,\, i+j,\, -i-j} + 2T_{i+j,\, i-j,\, i+j,\, i-j} \\ & + 2T_{i+j,\, -i+j,\, i+j,\, -i+j} + 2T_{i+j,\, -i-j,\, i-j,\, -i+j}]\, |B_{11}|^2 B_{11} \\ & + [2T_{i+j,\, 4i+4j,\, i+j,\, 4i+4j} + 2T_{i+j,\, -4i-4j,\, i+j,\, -4i-4j} + 2T_{i+j,\, 4i-4j,\, i+j,\, 4i-4j} \\ & + 2T_{i+j,\, -4i+4j,\, i+j,\, -4i+4j}]\, |B_{44}|^2 B_{11}, \end{aligned} \tag{6.4a}$$

$$\begin{aligned} \mathrm{i}\frac{\mathrm{d}B_{44}}{\mathrm{d}t} = {} & [2T_{4i+4j,\, i+j,\, 4i+4j,\, i+j} + 2T_{4i+4j,\, -i-j,\, 4i+4j,\, -i-j} \\ & + 2T_{4i+4j,\, i-j,\, 4i+4j,\, i-j} + 2T_{4i+4j,\, -i+j,\, 4i+4j,\, -i+j}]\, |B_{11}|^2 B_{44} \\ & + [T_{4i+4j,\, 4i+4j,\, 4i+4j,\, 4i+4j} + 2T_{4i+4j,\, -4i-4j,\, 4i+4j,\, -4i-4j} + 2T_{4i+4j,\, 4i-4j,\, 4i+4j,\, 4i-4j} \\ & + 2T_{4i+4j,\, -4i+4j,\, 4i+4j,\, -4i+4j} + 2T_{4i+4j,\, -4i-4j,\, 4i-4j,\, -4i+4j}]\, |B_{44}|^2 B_{44}. \end{aligned} \tag{6.4b}$$

Evaluation of the coefficients, combined with the substitutions (6.3 *b*), reduces (6.4 *a, b*) to

$$\mathrm{i}\frac{\mathrm{d}A_{11}}{\mathrm{d}t} = -\frac{45-8\sqrt{2}}{224}2^{1/4}|A_{11}|^2A_{11}+\frac{-19+5\sqrt{17}}{128}2^{1/4}|A_{44}|^2A_{11}, \quad (6.5a)$$

$$\mathrm{i}\frac{\mathrm{d}A_{44}}{\mathrm{d}t} = \frac{-19+5\sqrt{17}}{64}2^{1/4}|A_{11}|^2A_{44}-\frac{45-8\sqrt{2}}{7}2^{1/4}|A_{44}|^2A_{44}. \quad (6.5b)$$

Standing waves of Stokes type dominated by the first harmonic are given by (6.5 *a*) with

$$A_{11} = a_1\,\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}\phi_1}, \quad A_{44} = 0,$$

where a_1, ϕ_1, Ω are real constants, with the solution

$$\Omega = -\frac{45-8\sqrt{2}}{224}2^{1/4}a_1^2 = -0.150\,385\times 2^{1/4}a_1^2. \quad (6.6)$$

This frequency correction agrees with Verma & Keller (1962), equation (36), and with Okamura (1985), equations (2.8), (2.9). When it is rewritten as

$$\frac{\Omega}{2^{1/4}} = -\frac{45-8\sqrt{2}}{112}\left(\frac{a_1}{\sqrt{2}}\right)^2,$$

it is the same as (4.3 *c*), allowing for the rescaling of the frequency correction Ω and the slope parameter a_1.

Standing waves of the new type dominated by the first and fourth harmonics are given by (6.5 *a, b*) with

$$A_{11} = a_1\,\mathrm{e}^{-\mathrm{i}\Omega t+\mathrm{i}\phi_1}, \quad A_{44} = a_4\,\mathrm{e}^{-2\mathrm{i}\Omega t+\mathrm{i}\phi_4},$$

where $a_1, a_4, \phi_1, \phi_4, \Omega$ are real constants, with the solution

$$\Omega = -0.149\,535\times 2^{1/4}a_1^2, \quad \left|\frac{a_4}{a_1}\right| = 0.259\,600. \quad (6.7)$$

This is the same as (5.4 *b*) after allowing for rescaling.

First integrals of the evolution equations (6.5 *a, b*) show that the moduli $|A_{11}|$, $|A_{44}|$ and the rates of change of the arguments of A_{11}, A_{44} are all constant. This is a consequence of the absence of any resonant interaction causing the transfer of energy between the first and fourth harmonics to the tertiary order. The absence of energy transfer suggests that the standing waves are stable to other periodic disturbances also. The fully nonlinear evolution over 2000 wave periods was calculated with a small disturbance applied at wavenumber 1 in the x-direction; it was found that the three-dimensional waves of the Stokes type and of the new type both remained stable.

7. Discussion

The smallest possible frequency for standing waves in a square basin of side L is $(gk)^{1/2}$ rad s^{-1}, according to the linear theory, where $k = \pi L$. Linear theory predicts that there is an infinite number of standing wave solutions with this frequency. Nonlinear (or exact) theory shows that there are only a few stable standing wave solutions with constant frequencies near $(gk)^{1/2}$ rad s^{-1}. The free wave components of the stable standing wave solutions we have found, (4.3*c*, *d*) and (5.4*b*), rewritten in dimensional form with $\phi_i = 0$, are

$$\left.\begin{aligned} \eta &= a\cos kx\cos \omega t \pm a\cos ky\cos \omega t, \\ \omega &= (gk)^{1/2}(1-0.300\,77k^2a^2); \end{aligned}\right\} \tag{7.1a}$$

$$\left.\begin{aligned} \eta &= a\cos kx\cos \omega t \pm a\cos ky\sin \omega t, \\ \omega &= (gk)^{1/2}(1-0.074\,23k^2a^2); \end{aligned}\right\} \tag{7.1b}$$

$$\left.\begin{aligned} \eta &= a\cos kx\cos \omega t \pm a\cos ky\cos \omega t \pm 0.259\,60\,(a\cos 4kx\cos 2\omega t \pm a\cos 4ky\cos 2\omega t) \\ \omega &= (gk)^{1/2}(1-0.299\,07k^2a^2). \end{aligned}\right\} \tag{7.1c}$$

The solution (5.4*c*) is not included here because of inconsistencies with the fully nonlinear solution (5.4*f*, *g*). As far as we are aware, only (7.1*a*) has been described previously, rewritten in a formulation based on free wave components such as (6.1*a*).

For the three-dimensional standing waves when the slope parameter is 0.1, Okamura (1985, §3) concludes that three-dimensional disturbances applied to the wave given by (7.1*a*) above are unstable only on the resonance curves. His figure 2(*a*) shows no integer points on the curves, which is consistent with the stable behaviour in our nonlinear time evolution calculations of this wave.

All two-dimensional standing wave solutions are unstable to transverse disturbances, when cross-waves are initiated spontaneously in practice to grow to become comparable in magnitude with the initial standing waves. The energy transfer between the original waves and the cross-waves then reverses until the original waves return close to their initial state, and the cycle is repeated. This is the phenomenon of cyclic recurrence.

All three stable waves, given by (7.1*a*–*c*) are of equivalent 'importance' and one would expect to detect them in an appropriate experimental setting.

The overall very good agreement between the results obtained from the Zakharov equation and the full numerical solution is a clear refutation of the claim by Pierce & Knobloch (1994) that the Zakharov equation needs to be corrected when applied to standing waves.

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Addendum

My dear friend, Dr Peter Bryant, died suddenly of heart failure on November 25 1994, a few days after completing this paper. Peter Bryant was a mathematician who devoted his career to the study of water waves; a natural choice for a New Zealander, who grew up and lived near the ocean. His scientific works are published in leading international journals, a dozen of them in the Journal of Fluid Mechanics. I have had the privilege to collaborate with Peter during my recent sabbatical at the University of Canterbury and had been looking forward to his planned visit to Israel in April 1995. His early departure is a great loss, not only to his loving family and friends, but also to the international scientific community.

M.S.

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