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NOTE

The Induced Mean Flow Accompanying a Water-Wave Packet

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1. INTRODUCTION AND FORMULATION

The aim of this note is to provide quantitative information regarding the induced mean flow underneath a water-wave packet. Exact solutions of the cubic Schrödinger equation indicate that any initial wave packet eventually evolves into a number of envelope solitons and a dispersive tail. The bulk of the energy is contained in the solitons, which have solitary wave shapes and propagate with permanent form once produced. Only wave packets having the form of a single soliton are considered in the sequel. Following Dysthe (1979) one can show that the appropriate equations satisfied by the induced mean flow potential $\phi(x, z, t)$ are

$$\phi_{,xx} + \phi_{,zz} = 0, \qquad z \leqq 0, \quad -\infty < x < +\infty, \tag{1.1}$$

$$\phi_{,z} = \frac{a^2\omega}{2}\frac{\partial}{\partial x}\left\{\operatorname{sech}^2\left[\sqrt{2}ak^2\left(x - \frac{\omega}{2k}t\right)\right]\right\}, \quad z = 0, \tag{1.2}$$

$$\phi_{,z} = 0, \qquad z \to -\infty, \tag{1.3}$$

where a is the wave amplitude at the peak of the packet, ω and k are the frequency and wave number, related by the linear deep-water dispersion relation $\omega^2 = gk$.

The free surface elevation is given by

$$\eta = -\frac{1}{g}\phi_{,t} + a\operatorname{sech}\left[\sqrt{2}ak^2\left(x - \frac{\omega}{2k}t\right)\right] \cdot \cos\left[(kx - \omega t) - \frac{\omega}{4}a^2k^2t\right]. \tag{1.4}$$

Equations (1.1), (1.2) and (1.3) define a Neumann problem in the lower half plane.

2. SOLUTION

It seems helpful to use the following dimensionless variables:

$$X = ak^2\left(x - \frac{\omega}{2k}t\right); \qquad Z = ak^2 z; \qquad \Phi = \frac{1}{a^2\omega}\phi. \tag{2.1}$$

Substitution of Eq. (2.1) into Eqs. (1.1), (1.2) and (1.3) gives

$$\Phi_{,XX} + \Phi_{,ZZ} = 0, \qquad Z \leqq 0, \tag{2.2}$$

$$\Phi_{,Z} = \frac{1}{2}\frac{\partial}{\partial X}[\operatorname{sech}^2(\sqrt{2}X)], \qquad Z = 0, \tag{2.3}$$

$$\Phi_{,Z} = 0, \qquad Z \to -\infty. \tag{2.4}$$

The above problem was solved utilizing the Fourier transform method, with the following result:

$$\Phi = X \cdot \sum_{n=0}^{\infty} (Z - \alpha_n)/[X^2 + (Z - \alpha_n)^2]^2 \tag{2.5}$$

where

$$\alpha_n = \frac{\pi}{\sqrt{2}}\left(\tfrac{1}{2} + n\right). \tag{2.6}$$

A reader who would like to check this solution (by substituting it back into the set of Eqs. (2.2), (2.3) and (2.4)) will find the following identity helpful:

$$\operatorname{sech}^2\zeta = -2\sum_{n=0}^{\infty} (\zeta^2 - 2\alpha_n^2)/(\zeta^2 + 2\alpha_n^2)^2; \tag{2.7}$$

see Eq. (3.64) in Carrier et al. (1966).

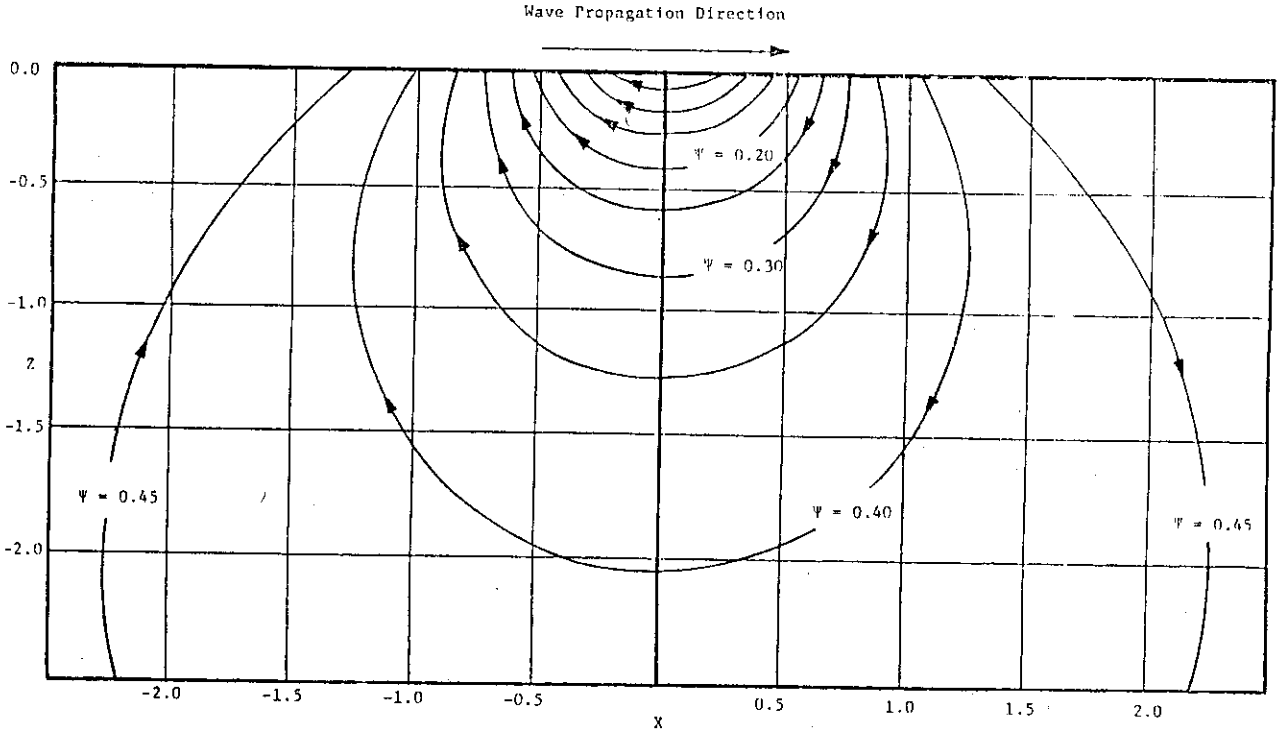

Fig. 1. The wave-induced mean flow field.

3. RESULTS AND DISCUSSION

Let u and v be the induced mean flow velocity components in the directions x and z, respectively. From Eq. (2.5) and Eq. (2.1) one can show that

$$U = \frac{u}{(a\omega)(ak)^2} = \sum_{n=0}^{\infty}(Z-\alpha_n)[(Z-\alpha_n)^2 - 3X^2]/[X^2+(Z-\alpha_n)^2]^3, \tag{3.1}$$

$$V = \frac{v}{(a\omega)(ak)^2} = X \cdot \sum_{n=0}^{\infty}[X^2 - 3(Z-\alpha_n)^2]/[X^2+(Z-\alpha_n)^2]^3. \tag{3.2}$$

The dimensionless stream function is given by

$$\Psi(X,Z) = \frac{1}{a^2\omega}\psi = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{X^2-(Z-\alpha_n)^2}{[X^2+(Z-\alpha_n)^2]^2}. \tag{3.3}$$

Note that $\Sigma_{n=0}^{\infty}\alpha_n^{-2} = 1$ and that the first term on the r.h.s. of Eq. (3.3) was chosen to render $\Psi(0,0) = 0$. The streamlines of the induced mean flow field are shown in Fig. 1. The total flux, per unit width, involved in this flow is given by

$$q = \psi(\infty,0) - \psi(0,0) = a^2\omega[\Psi(\infty,0) - \Psi(0,0)] = \tfrac{1}{2}a^2\omega. \tag{3.4}$$

This value is equal to the Stokesian mass transport at the peak of the wave packet, as it should be.

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