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On modifications of the Zakharov equation for surface gravity waves

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The Zakharov integral equation for surface gravity waves is modified to include higher-order (quintet) interactions, for water of constant (finite or infinite) depth. This new equation is used to study some aspects of class I (4-wave) and class II (5-wave) instabilities of a Stokes wave.

1. Introduction

Our understanding of the nonlinear dynamics of deep-water gravity waves has grown substantially in recent years. We feel that the lion's share of this progress should be attributed to the staff of the TRW Fluid Mechanics Department. Most of their findings are summarized in an extensive review article by Yuen & Lake (1982), which served as our main reference. Much of this progress is based on applications of the so-called Zakharov equation which was originally derived by Zakharov (1968) for infinitely deep water. Zakharov & Kharitonov (1970) extended the derivation to arbitrary water depth, but didn't present the interaction coefficients. For the sake of a comprehensive discussion, we re-derive Zakharov's equation for finite water depth (in §2) and show its relations to the cubic Schrödinger equation and to Hasselmann's nonlinear interaction model (in §3). It is our opinion that the Zakharov equation is superior to all other existing approximate models as far as class I interactions are concerned.

The term 'class I interactions' refers to nonlinear interaction processes at the lowest possible order; for surface gravity waves this occurs at third order in the nonlinearity parameter ϵ. Generally speaking, class I interactions require the coexistence of resonating, or nearly resonating, wave quartets. The time scale of class I interactions is $\epsilon^{-2} P$ where P is a typical wave period.

The structure of the surface-gravity-wave dispersion relation does not enable nonlinear interaction at shorter timescales ($\epsilon^{-1} P$) which occur in many other physical systems (e.g. capillary waves).

While class I interactions are basically four-wave interactions, the special case where one of the waves is taken into account twice so that only three waves are considered has attracted much attention. These cases which lead to what sometimes is called Benjamin–Feir instabilities, display many of the features of the more general quartet interaction. Interactions including a smaller number of waves – as two waves each taken into account twice, or one wave taken into account four times – are also possible, but display a degenerated type of interaction which manifests itself in Stokes-type second-order corrections of the frequency (see Longuet-Higgins &

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Phillips 1962). Numerical linear stability analysis of the exact finite-amplitude Stokes wave, by McLean (1982*a*, *b*), as well as experimental evidence by Su *et al.* (1982) and Su (1982), reveal the importance of class II interactions, which are basically quintet interactions. These, much-less studied interactions, occur at fourth order in ϵ and have a typical timescale of $\epsilon^{-3}P$. Nevertheless, for high-enough steepnesses McLean's study, as well as the earlier work of Longuet-Higgins (1978), show that class II instabilities become dominant. Here again, three waves – one of them is taken into account three times – form a nearly resonating quintet and display many interesting features. In the second half of §2 we extend the derivation to fourth order and derive a modified form of the Zakharov equation which accounts for both class I and the higher-order, class II, interactions.

In §4 we use this equation to study the linear stability of a uniform wavetrain. The solution of certain long-time evolution problems is under way and will be reported at a later stage.

2. The governing equations

The equations governing the irrotational flow of an incompressible inviscid fluid with a free surface are

$$\nabla^2\phi = 0 \quad (-h \leqslant z \leqslant \eta(\boldsymbol{x},t)), \tag{2.1}$$

$$\left.\begin{aligned} \eta_t + (\nabla\phi)\cdot(\nabla\eta) - \phi_z &= 0, \\ \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz &= 0, \end{aligned}\right\} \quad (z = \eta(\boldsymbol{x},t)), \tag{2.2a, b}$$

$$\phi_z = 0 \quad (z = -h), \tag{2.3}$$

where ϕ is the velocity potential, η is the free surface and g is the gravitational acceleration. The horizontal coordinates are $(x_1, x_2) = \boldsymbol{x}$, the vertical coordinate z is pointing upwards, h is the mean water depth, and t is the time.

The free-surface boundary conditions (2.2) are rewritten in terms of ϕ^{s} and $w^{\mathrm{s}} = (\partial\phi/\partial z)|_{z=\eta}$, the velocity potential and the vertical velocity component at the free surface, respectively:

$$\eta_t + (\nabla_x\phi^{\mathrm{s}})\cdot(\nabla_x\eta) - w^{\mathrm{s}}[1+(\nabla_x\eta)^2] = 0, \tag{2.4a}$$

$$\phi^{\mathrm{s}}_t + g\eta + \tfrac{1}{2}(\nabla_x\phi^{\mathrm{s}})^2 - \tfrac{1}{2}(w^{\mathrm{s}})^2[1+(\nabla_x\eta)^2] = 0. \tag{2.4b}$$

The horizontal Fourier transform of these equations yields

$$\begin{aligned} \hat{\eta}_t(\boldsymbol{k},t) &- \frac{1}{2\pi}\iint_{-\infty}^{\infty} (\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\,\hat{\phi}^{\mathrm{s}}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 - \hat{w}^{\mathrm{s}} \\ &+ \frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty} (\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\,\hat{w}^{\mathrm{s}}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \\ &\times \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 = 0, \end{aligned} \tag{2.5a}$$

$$\begin{aligned} \hat{\phi}^{\mathrm{s}}_t(\boldsymbol{k},t) + g\hat{\eta}(\boldsymbol{k},t) &- \frac{1}{4\pi}\iint_{-\infty}^{\infty} (\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\,\hat{\phi}^{\mathrm{s}}(\boldsymbol{k}_1,t)\,\hat{\phi}^{\mathrm{s}}(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 \\ - \frac{1}{4\pi}\iint_{-\infty}^{\infty} &\hat{w}^{\mathrm{s}}(\boldsymbol{k}_1,t)\,\hat{w}^{\mathrm{s}}(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 \\ + \frac{1}{16\pi^3}\iiiint_{-\infty}^{\infty} &(\boldsymbol{k}_3\cdot\boldsymbol{k}_4)\,\hat{w}^{\mathrm{s}}(\boldsymbol{k}_1,t)\,\hat{w}^{\mathrm{s}}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t) \\ &\times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4, \end{aligned} \tag{2.5b}$$

where the two-dimensional Fourier transform of a function $f(\boldsymbol{x})$ is given by

$$\hat{f}(\boldsymbol{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x},$$

and the Dirac δ-function is defined as

$$\delta(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}.$$

Taking the Fourier transform of the Laplace equation (2.1), and satisfying the boundary condition (2.3) at the bottom gives

$$\hat{\phi}(\boldsymbol{k}, z, t) = \hat{\Phi}(\boldsymbol{k}, t)\,\mathrm{ch}\,(|\boldsymbol{k}|\,(z+h)), \tag{2.6}$$

which enables one to write ϕ^{s} and w^{s} in terms of $\hat{\Phi}(\boldsymbol{k},t)$ and $\eta(\boldsymbol{x},t)$ as follows:

$$\phi^{\mathrm{s}}(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\Phi}(\boldsymbol{k},t)\,[\mathrm{ch}\,(|\boldsymbol{k}|\,h)\,\mathrm{ch}\,(|\boldsymbol{k}|\,\eta(\boldsymbol{x},t)) + \mathrm{sh}\,(|\boldsymbol{k}|\,h)\,\mathrm{sh}\,(|\boldsymbol{k}|\,\eta(\boldsymbol{x},t))]\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}, \tag{2.7a}$$

$$w^{\mathrm{s}}(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\boldsymbol{k}|\,\hat{\Phi}(\boldsymbol{k},t)\,[\mathrm{ch}\,(|\boldsymbol{k}|\,h)\,\mathrm{sh}\,(|\boldsymbol{k}|\,\eta(\boldsymbol{x},t)) + \mathrm{sh}\,((|\boldsymbol{k}|\,h)\,\mathrm{ch}\,(|\boldsymbol{k}|\,\eta(\boldsymbol{x},t))]\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k}. \tag{2.7b}$$

The next step in the derivation is to express $\hat{w}^{\mathrm{s}}$ as a function of $\hat{\eta}$ and $\hat{\phi}^{\mathrm{s}}$. This is the first step which requires an additional physical assumption. Assuming that $|\boldsymbol{k}|\,\eta$ is small, we pursue the following procedure: (i) replace sh $(|\boldsymbol{k}|\,\eta)$ and ch $(|\boldsymbol{k}|\,\eta)$, in (2.7*a, b*), by their Taylor-series expansions up to order $(|\boldsymbol{k}|\,\eta)^3$; (ii) express η by means of its Fourier transform $\hat{\eta}$; and finally (iii) take the Fourier transform of (2.7*a, b*):

$$\begin{aligned}\hat{\phi}^{\mathrm{s}}(\boldsymbol{k},t) = {} & \hat{\Phi}(\boldsymbol{k},t)\,\mathrm{ch}\,(|\boldsymbol{k}|\,h) + \frac{1}{2\pi}\iint_{-\infty}^{\infty} |\boldsymbol{k}_1|\,\mathrm{sh}\,(|\boldsymbol{k}_1|\,h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t) \\ & \times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 + \frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty} \tfrac{1}{2}|\boldsymbol{k}_1|^2\,\mathrm{ch}\,(|\boldsymbol{k}_1|\,h) \\ & \times \hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 \\ & + \frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty} \tfrac{1}{6}|\boldsymbol{k}_1|^3\,\mathrm{sh}\,(|\boldsymbol{k}_1|\,h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t) \\ & \times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4,\end{aligned} \tag{2.8a}$$

$$\begin{aligned}\hat{w}^{\mathrm{s}}(\boldsymbol{k},t) = {} & |\boldsymbol{k}|\,\hat{\Phi}(\boldsymbol{k},t)\,\mathrm{sh}\,(|\boldsymbol{k}|\,h) \\ & + \frac{1}{2\pi}\iint_{-\infty}^{\infty} |\boldsymbol{k}_1|^2\,\mathrm{ch}\,(|\boldsymbol{k}_1|\,h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 \\ & + \frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty} \tfrac{1}{2}|\boldsymbol{k}_1|^3\,\mathrm{sh}\,(|\boldsymbol{k}_1|\,h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3 \\ & + \frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty} \tfrac{1}{6}|\boldsymbol{k}_1|^4\,\mathrm{ch}\,(|\boldsymbol{k}_1|\,h)\,\hat{\Phi}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t) \\ & \times \delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4.\end{aligned} \tag{2.8b}$$

Inverting (2.8a) iteratively, in order to obtain $\hat{\Phi} = \hat{\Phi}(\hat{\phi}^{s})$, and substituting the result into (2.8b), yields

$$\hat{w}^{s}(\boldsymbol{k},t) = |\boldsymbol{k}|\,\mathrm{th}\,(|\boldsymbol{k}|\,h)\,\hat{\phi}^{s}(\boldsymbol{k},t)$$
$$-\frac{1}{2\pi}\iint_{-\infty}^{\infty}|\boldsymbol{k}_1|\,[|\boldsymbol{k}|\,\mathrm{th}\,((|\boldsymbol{k}|\,h)\,\mathrm{th}\,(|\boldsymbol{k}|\,h)-|\boldsymbol{k}_1|]\,\hat{\phi}^{s}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2$$
$$-\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}S^{(1)}(\boldsymbol{k},\boldsymbol{k}_1\,\boldsymbol{k}_2,\boldsymbol{k}_3)\,\hat{\phi}^{s}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3$$
$$-\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}S^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\,\hat{\phi}^{s}(\boldsymbol{k}_1,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4\,t)\,\hat{\eta}(\boldsymbol{k}_2,t)\,\hat{\eta}(\boldsymbol{k}_3,t)\,\hat{\eta}(\boldsymbol{k}_4,t)$$
$$\times\,\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4. \qquad (2.9)$$

The kernels $S^{(1)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$, $S^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)$, as well as other kernels, which appear throughout the derivation, are given in the Appendix. Substituting $\hat{w}^{s}$ from (2.9) into (2.5a, b), multiplying (2.5a) by $[g/(2\omega(\boldsymbol{k})]^{\frac{1}{2}}$, and (2.5$b$) by $\mathrm{i}[\omega(\boldsymbol{k})/(2g)]^{\frac{1}{2}}$, where

$$\omega(\boldsymbol{k}) = [g\,|\boldsymbol{k}|\,\mathrm{th}\,(|\boldsymbol{k}|\,h)]^{\frac{1}{2}}, \qquad (2.10)$$

adding these equations together, and defining the new complex variable

$$b(\boldsymbol{k},t) = \left[\frac{g}{2\omega(\boldsymbol{k})}\right]^{\frac{1}{2}}\hat{\eta}(\boldsymbol{k},t)+\mathrm{i}\left[\frac{\omega(\boldsymbol{k})}{2g}\right]^{\frac{1}{2}}\hat{\phi}^{s}(\boldsymbol{k},t),$$

yields the following equation:

$$b_t(\boldsymbol{k},t)+\mathrm{i}\omega(\boldsymbol{k})\,b(\boldsymbol{k},t)+\mathrm{i}\sum_{n=1}^{3}\iint_{-\infty}^{\infty}V^{(n)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)\,C_{2n}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2$$
$$+\mathrm{i}\sum_{n=1}^{4}\iiint_{-\infty}^{\infty}W^{(n)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,C_{3n}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3$$
$$+\mathrm{i}\sum_{n=1}^{5}\iiiint_{-\infty}^{\infty}X^{(n)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)\,C_{4n}\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3\,\mathrm{d}\boldsymbol{k}_4 = 0, \qquad (2.11)$$

where

$$C_{ln} = \left(\prod_{m=1}^{n-1}b^{*}(\boldsymbol{k}_m,t)\right)\left(\prod_{m=n}^{l}b(\boldsymbol{k}_m,t)\right)\delta\left(\boldsymbol{k}+\sum_{m=1}^{n-1}\boldsymbol{k}_m-\sum_{m=n}^{l}\boldsymbol{k}_m\right),$$

where $*$ denotes the complex conjugate, and $\Sigma_{m=n}^{l}$, $\Pi_{m=n}^{l}$ for $l < n$ are defined to be 0 and 1 respectively.

The relations between η, ϕ^{s} and the complex 'amplitude spectrum' b are

$$\hat{\eta}(\boldsymbol{k},t) = \left|\frac{\omega(\boldsymbol{k})}{2g}\right|^{\frac{1}{2}}[b(\boldsymbol{k},t)+b^{*}(-\boldsymbol{k},t)], \qquad (2.12a)$$

$$\hat{\phi}^{s}(\boldsymbol{k},t) = -\mathrm{i}\left|\frac{g}{2\omega(\boldsymbol{k})}\right|^{\frac{1}{2}}[b(\boldsymbol{k},t)-b^{*}(-\boldsymbol{k},t)]. \qquad (2.12b)$$

We assume that the wave field can be divided into a slowly varying (in time) component B and small rapidly varying components B', B'', B''' and that most of the energy in the wave field is contained in B. These assumptions permit one to write

$$b(\boldsymbol{k},t) = [\epsilon\tilde{B}(\boldsymbol{k},t_2,t_3)+\epsilon^2B'(\boldsymbol{k},t,t_2,t_3)+\epsilon^2B''(\boldsymbol{k},t,t_2,t_3)$$
$$+\epsilon^4B'''(\boldsymbol{k},t,t_2,t_3)]\,\mathrm{e}^{-\mathrm{i}\omega(\boldsymbol{k})\,t}, \qquad (2.13)$$

where ϵ is a small parameter representing the magnitude of nonlinearity, and the slow timescales are defined by $t_2 = \epsilon^2 t$, $t_3 = \epsilon^3 t$. The omission of the slow time $t_1 = \epsilon t$ from (2.13) results from the fact that resonating triads do not exist for surface gravity waves. Substituting b, from (2.13), into (2.11) and arranging the terms according to their order in ϵ yields the following results.

Order ϵ is satisfied identically.

Order ϵ^2 gives an equation for B':

$$\mathrm{i}\frac{\partial B'}{\partial t} = \iint_{-\infty}^{\infty} \Big\{ V^{(1)}_{0,1,2} \tilde{B}_1 \tilde{B}_2 \delta_{0-1-2} \mathrm{e}^{\mathrm{i}(\omega-\omega_1-\omega_2)t} + V^{(2)}_{0,1,2} \tilde{B}^*_1 \tilde{B}_2 \delta_{0+1-2} \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2)t} + V^{(3)}_{0,1,2} \tilde{B}^*_1 \tilde{B}^*_2 \delta_{0+1+2} \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2)t} \Big\} \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2, \quad (2.14a)$$

where we have introduced a compact notation in which the arguments $\boldsymbol{k}_i$ in V, $\tilde{B}$, δ, ω and in other functions in the sequel are replaced by subscripts i, with the subscript zero assigned to $\boldsymbol{k}$. Integrating (2.14a) with respect to t and keeping t_2, t_3 fixed gives

$$B' = -\iint_{-\infty}^{\infty} \Big\{ V^{(1)}_{0,1,2} \tilde{B}_1 \tilde{B}_2 \delta_{0-1-2} \frac{\mathrm{e}^{\mathrm{i}(\omega-\omega_1-\omega_2)t}}{\omega-\omega_1-\omega_2} + V^{(2)}_{0,1,2} \tilde{B}^*_1 \tilde{B}_2 \delta_{0+1-2} \frac{\mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2)t}}{\omega+\omega_1-\omega_2} + V^{(3)}_{0,1,2} \tilde{B}^*_1 \tilde{B}^*_2 \delta_{0+1+2} \frac{\mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2)t}}{\omega+\omega_1+\omega_2} \Big\} \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2. \quad (2.14b)$$

The constant of integration, which corresponds to the initial phase, has been set to zero without loss of generality.

Order ϵ^3 gives the following equation:

$$\mathrm{i}\frac{\partial \tilde{B}}{\partial t_2} + \mathrm{i}\frac{\partial B''}{\partial t} = \iiint_{-\infty}^{\infty} \Big\{ \tilde{T}^{(1)}_{0,1,2,3} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \delta_{0-1-2-3} \mathrm{e}^{\mathrm{i}(\omega-\omega_1-\omega_2-\omega_3)t} + \tilde{T}^{(2)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t} + \tilde{T}^{(3)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \delta_{0+1+2-3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2-\omega_3)t} + \tilde{T}^{(4)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \delta_{0+1+2+3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2+\omega_3)t} \Big\} \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3. \quad (2.15)$$

The above equation consists of terms of two types; those that depend on the fast time t, and those that do not. This enables us to split (2.15) into separate equations:

$$\mathrm{i}\frac{\partial \tilde{B}}{\partial t_2} = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t} \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3, \quad (2.16)$$

$$\mathrm{i}\frac{\partial B''}{\partial t} = \iiint_{-\infty}^{\infty} \Big\{ \tilde{T}^{(1)}_{0,1,2,3} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \delta_{0-1-2-3} \mathrm{e}^{\mathrm{i}(\omega-\omega_1-\omega_2-\omega_3)t} + (\tilde{T}^{(2)}_{0,1,2,3} - T^{(2)}_{0,1,2,3}) \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t} + \tilde{T}^{(3)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \delta_{0+1+2-3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2-\omega_3)t} + \tilde{T}^{(4)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \delta_{0+1+2+3} \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2+\omega_3)t} \Big\} \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3. \quad (2.17a)$$

Here we have made use of the fact that the only exponent of e that may become zero, under the restriction of the δ-functions, is the one in the second term in the right-hand side of (2.15). This fact is directly related to the definition of a nearly resonating quartet:

$$\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3 = 0, \quad |\omega+\omega_1-\omega_2-\omega_3| \leqslant O(\epsilon^2). \qquad (2.18a, b)$$

Equation (2.16) is the so-called Zakharov equation, with the kernel

$$T^{(2)}_{0,1,2,3} = \begin{cases} \tilde{T}^{(2)}_{0,1,2,3} & \text{for near resonance quartets,} \\ 0 & \text{otherwise,} \end{cases} \qquad (2.19)$$

used as a mathematical model for class I nonlinear interactions. Integrating (2.17a) with respect to t gives the following result for B'':

$$\begin{aligned} B'' = -\iiint_{-\infty}^{\infty} \Big\{ & \tilde{T}^{(1)}_{0,1,2,3} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \delta_{0-1-2-3} \frac{e^{i(\omega-\omega_1-\omega_2-\omega_3)t}}{\omega-\omega_1-\omega_2-\omega_3} \\ & + (\tilde{T}^{(2)}_{0,1,2,3} - T^{(2)}_{0,1,2,3}) \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} \frac{e^{i(\omega+\omega_1-\omega_2-\omega_3)t}}{\omega+\omega_1-\omega_2-\omega_3} \\ & + \tilde{T}^{(3)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \delta_{0+1+2-3} \frac{e^{i(\omega+\omega_1+\omega_2-\omega_3)t}}{\omega+\omega_1+\omega_2-\omega_3} \\ & + \tilde{T}^{(4)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \delta_{0+1+2+3} \frac{e^{i(\omega+\omega_1+\omega_2+\omega_3)t}}{\omega+\omega_1+\omega_2+\omega_3} \Big\} d\boldsymbol{k}_1 \, d\boldsymbol{k}_2 \, d\boldsymbol{k}_3. \end{aligned} \qquad (2.17b)$$

Order ϵ^4:

$$\begin{aligned} i\frac{\partial \tilde{B}}{\partial t_3} + i\frac{\partial B'''}{\partial t} = -i\frac{\partial B'}{\partial t_2} + \iiiint_{-\infty}^{\infty} \Big\{ & \tilde{U}^{(1)}_{0,1,2,3,4} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \tilde{B}_4 \delta_{0-1-2-3-4} e^{i(\omega-\omega_1-\omega_2-\omega_3-\omega_4)t} \\ & + \tilde{U}^{(2)}_{0,1,2,3,4} \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \tilde{B}_4 \delta_{0+1-2-3-4} e^{i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)t} \\ & + \tilde{U}^{(3)}_{0,1,2,3,4} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \tilde{B}_4 \delta_{0+1+2-3-4} e^{i(\omega+\omega_1+\omega_2-\omega_3-\omega_4)t} \\ & + \tilde{U}^{(4)}_{0,1,2,3,4} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \tilde{B}_4 \delta_{0+1+2+3-4} e^{i(\omega+\omega_1+\omega_2+\omega_3-\omega_4)t} \\ & + \tilde{U}^{(5)}_{0,1,2,3,4} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \tilde{B}^*_4 \delta_{0+1+2+3+4} e^{i(\omega+\omega_1+\omega_2+\omega_3+\omega_4)t} \Big\} d\boldsymbol{k}_1 \, d\boldsymbol{k}_2 \, d\boldsymbol{k}_3 \, d\boldsymbol{k}_4. \end{aligned} \qquad (2.20)$$

In order to split (2.20) appropriately into two separate equations, one for $\partial\tilde{B}/\partial t_3$ and the other for $\partial B'''/\partial t$, which becomes relevant only in sextet interactions, we make use of the fact that only the second and third integrands in (2.20) enable resonating quintets. Similarly to (2.18), the nearly resonating quintets are defined by

$$\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3-\boldsymbol{k}_4 = 0, \quad |\Omega+\Omega_1-\Omega_2-\Omega_3-\Omega_4| \leqslant O(\epsilon^3), \qquad (2.21a, b)$$

where Ω_j, the 'Stokes-corrected' frequencies, are given by

$$\Omega_j = \omega_j + \epsilon^2 \int_{-\infty}^{\infty} e_{j1} T_{j1j1} |\tilde{B}_1|^2 \, d\boldsymbol{k}_1, \quad e_{j1} = \begin{cases} 2 & (j \neq 1), \\ 1 & (j = 1). \end{cases} \qquad (2.21c)$$

The Stokes corrected frequencies are obtained by solving (2.16) for degenerated interactions, namely 'quartets' formed by two waves, each taken into account twice. These corrections become necessary at the order of derivation considered here.

Defining

$$U^{(2)}_{0,1,2,3,4} = \begin{cases} \tilde{U}^{(2)}_{0,1,2,3,4} & \text{for nearly resonating quintets,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.22a}$$

$$U^{(3)}_{0,1,2,3,4} = \begin{cases} \tilde{U}^{(3)}_{0,1,2,3,4} & \text{for nearly resonating quintets,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.22b}$$

we obtain

$$\mathrm{i}\frac{\partial \tilde{B}}{\partial t_3} = \iiiint_{-\infty}^{\infty} \Big\{ U^{(2)}_{0,1,2,3,4} \tilde{B}_1^* \tilde{B}_2 \tilde{B}_3 \tilde{B}_4 \delta_{0+1-2-3-4} \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3-\omega_4)t}$$
$$+ U^{(3)}_{0,1,2,3,4} \tilde{B}_1^* \tilde{B}_2^* \tilde{B}_3 \tilde{B}_4 \delta_{0+1+2-3-4} \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2-\omega_3-\omega_4)t} \Big\} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3 \, \mathrm{d}\boldsymbol{k}_4. \tag{2.23}$$

Finally, the two orders, (2.16) and (2.23), are combined into a single equation for $B = \epsilon \tilde{B}$:

$$\mathrm{i}\frac{\partial B}{\partial t} = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3} B_1^* B_2 B_3 \delta_{0+1-2-3} \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3$$
$$+ \iiiint_{-\infty}^{\infty} U^{(2)}_{0,1,2,3,4} B_1^* B_2 B_3 B_4 \delta_{0+1-2-3-4} \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3-\omega_4)t} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3 \, \mathrm{d}\boldsymbol{k}_4$$
$$+ \iiiint_{-\infty}^{\infty} U^{(3)}_{0,1,2,3,4} B_1^* B_2^* B_3 B_4 \delta_{0+1+2-3-4} \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1+\omega_2-\omega_3-\omega_4)t} \, \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3 \, \mathrm{d}\boldsymbol{k}_4. \tag{2.24}$$

Equation (2.24) is a modification of the Zakharov equation that accounts for higher-order interactions.

3. Comparison of the Zakharov equation with other model equations

Denoting $\epsilon \tilde{B}$ by B, we rewrite (2.16) as

$$\mathrm{i}\frac{\partial B(\boldsymbol{k},t)}{\partial t} = \iiint_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k}_1 \, \mathrm{d}\boldsymbol{k}_2 \, \mathrm{d}\boldsymbol{k}_3 \, T^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) B^*(\boldsymbol{k}_1,t) B(\boldsymbol{k}_2,t) B(\boldsymbol{k}_3,t)$$
$$\times \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \, \mathrm{e}^{\mathrm{i}(\omega+\omega_1-\omega_2-\omega_3)t} \tag{3.1}$$

The first-order free-surface elevation is related to B through (2.12*a*) and (2.13), and is given by

$$\eta(\boldsymbol{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\omega(\boldsymbol{k})}{2g}\right)^{\frac{1}{2}} \{B(\boldsymbol{k},t)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} + \text{c.c.}\} \, \mathrm{d}\boldsymbol{k}. \tag{3.2}$$

Equation (3.1) is the now well-known Zakharov equation, generalized for water of any constant depth. The fact that (3.1) is valid for finite depth affects only the expressions for $\omega(\boldsymbol{k})$ and $T^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$, which become depth-dependent.

The purpose of this section is to show the connections between the Zakharov equation and other model equations, as well as to check our depth-dependent expression for $T^{(2)}$. Note that for $h \to \infty$ our equation (A 5*c*) for $T^{(2)}$ gives the same result as does Appendix A of Crawford *et al.* (1981). This result is different from that given in Yuen & Lake (1982) (even after corrections of minor misprints). This apparent discrepancy is related to the special, almost-symmetric (with respect to $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$) structure of the Zakharov equation. This structure allows some freedom in the choice of $T^{(2)}$, i.e. $T^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ can be replaced by $\alpha T^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)+(1-\alpha)\,T^{(2)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_3,\boldsymbol{k}_2)$, with arbitrary α, without altering the value of the integral on the right-hand side of (3.1). Any $T^{(2)}$, obtained in some legitimate derivation, can be made symmetric in $\boldsymbol{k}_2$, $\boldsymbol{k}_3$ by choosing $\alpha = 0.5$. This symmetric $T^{(2)}$, denoted by T, is a uniquely defined function of $\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3$ and h and will be used in the sequel.

3.1. *Relation to Hasselmann's energy-transfer model*

The energy-transfer equation for a finite-depth gravity-wave spectrum, originally obtained in Hasselmann (1962), was rederived by Herterich & Hasselmann (1980). This later paper served as a reference for the verification of the expression (A 5*c*) for T. By reasonings similar to those in Longuet-Higgins (1976), but starting from the Zakharov equation (instead of the cubic Schrödinger equation, used by Longuet-Higgins) the following energy-transfer equation is obtained:

$$\frac{\partial C(\boldsymbol{k},t)}{\partial t} = 4\pi \iiint_{-\infty}^{\infty} T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\,[C(\boldsymbol{k}_2)\,C(\boldsymbol{k}_3)\,(C(\boldsymbol{k})+C(\boldsymbol{k}_1))-C(\boldsymbol{k})\,C(\boldsymbol{k}_1)\,(C(\boldsymbol{k}_2) + C(\boldsymbol{k}_3))]\,\delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)\,\delta(\omega+\omega_1-\omega_2-\omega_3)\,\mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_3, \quad (3.3)$$

where the wave-action spectrum $C = |B|^2$.

For strict resonance conditions, which are implied by the two δ-functions in (3.3), T is also symmetric in its two first arguments $\boldsymbol{k}$ and $\boldsymbol{k}_1$. Herterich & Hasselmann's $F(\boldsymbol{k},t)$ is given by $\omega(\boldsymbol{k})\,C(\boldsymbol{k},t)/4g\pi^2$, and their interaction coefficient D is given by

$$D(\boldsymbol{k}_3,\boldsymbol{k}_2,\boldsymbol{k}_1) \equiv -\frac{16\pi^2}{3g}(\omega\omega_1\,\omega_2\,\omega_3)^{\frac{1}{2}}\,T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) \quad (3.4)$$

for resonating quartets.

The above identity (3.4) has been verified numerically, and thus serves as a mutual check of the rather lengthy algebra involved in the derivation of both models.

3.2. *Relation to the nonlinear Schrödinger equation*

The derivation here follows the lines of Zakharov (1968), who showed that, in the case of infinitely deep water, the cubic Schrödinger equation is a particular case of the more general Zakharov equation. In the case of finite water depth, the value of $T(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$, in the limit when $\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3$ tend to $\boldsymbol{k}$ and h is fixed is not unique. In order to provide a better grasp of this nonuniqueness, we include here an outline of the derivation of the finite-depth nonlinear Schrödinger equation.

Restricting the analysis to narrow spectra around $\boldsymbol{k}_0 = (k_0, 0)$, we rewrite all wavenumbers as $\boldsymbol{k}_i = \boldsymbol{k}_0+\boldsymbol{\psi}_i$, $\boldsymbol{\psi}_i = (\psi_i, \lambda_i)$ and $|\boldsymbol{\psi}|/k_0 \ll 1$. Introducing a new variable $A(\boldsymbol{\psi},t) = B(\boldsymbol{k},t)\,\mathrm{e}^{-\mathrm{i}[\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_0)]t}$ into (3.1) gives

$$\mathrm{i}\frac{\partial A(\boldsymbol{\psi},t)}{\partial t} - [\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_0)]\,A(\boldsymbol{\psi},t) = \iiint_{-\infty}^{\infty} T(\boldsymbol{k}_0+\boldsymbol{\psi},\boldsymbol{k}_0+\boldsymbol{\psi}_1,\boldsymbol{k}_0+\boldsymbol{\psi}_2,\boldsymbol{k}_0+\boldsymbol{\psi}_3)\,A^*(\boldsymbol{\psi}_1)\,A(\boldsymbol{\psi}_2)\,A(\boldsymbol{\psi}_3) \times \delta(\boldsymbol{\psi}+\boldsymbol{\psi}_1-\boldsymbol{\psi}_2-\boldsymbol{\psi}_3)\,\mathrm{d}\boldsymbol{\psi}_1\,\mathrm{d}\boldsymbol{\psi}_2\,\mathrm{d}\boldsymbol{\psi}_3. \quad (3.5)$$

Equation (3.2) is then expanded to the lowest order in the spectral width:

$$\eta(\boldsymbol{x},t) = \frac{1}{2\pi}\left(\frac{\omega(\boldsymbol{k}_0)}{2g}\right)^{\frac{1}{2}}\left\{e^{i[k_0 x_1-\omega(k_0)t]}\int_{-\infty}^{\infty} A(\boldsymbol{\psi},t)\,e^{i\boldsymbol{\psi}\cdot\boldsymbol{x}}\,d\boldsymbol{\psi}+\text{c.c.}\right\}$$
$$= \operatorname{Re}\{a(\boldsymbol{x},t)\,e^{i[k_0x_1-\omega(k_0)t]}\}, \tag{3.6}$$

where the complex wave envelope $a(\boldsymbol{x},t)$ is the inverse Fourier transform of $A(\boldsymbol{\psi},t)$, multiplied by the coefficient $(2\omega(k_0)/g)^{\frac{1}{2}}$. The frequency difference on the left-hand side of (3.5) is replaced by its Taylor-series expansion up to the second order in the spectral width:

$$\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_0) = c_g\,\psi+\frac{c_g}{2k_0}\lambda^2+\tfrac{1}{2}c_g'\psi^2+O(|\boldsymbol{\psi}|^3), \tag{3.7}$$

where $c_g = \partial\omega_0/\partial k_0$, $c_g' = \partial^2\omega_0/\partial k_0^2$ and $\omega_0^2 = gk_0\,\text{th}\,(k_0 h)$. Multiplying (3.5) by $(2\omega_0/g)^{\frac{1}{2}}$ and taking its inverse Fourier transform yields

$$i\left(\frac{\partial a}{\partial t}+c_g\frac{\partial a}{\partial x_1}\right)+\frac{c_g}{2k_0}\frac{\partial^2 a}{\partial x_2^2}+\frac{c_g'}{2}\frac{\partial^2 a}{\partial x_1^2}$$
$$= \frac{1}{2\pi}\left(\frac{2\omega_0}{g}\right)^{\frac{1}{2}}\iiint_{-\infty}^{\infty} T(\boldsymbol{k}_0+\boldsymbol{\psi}_2+\boldsymbol{\psi}_3-\boldsymbol{\psi}_1,\boldsymbol{k}_0+\boldsymbol{\psi}_1,\boldsymbol{k}_0+\boldsymbol{\psi}_2,\boldsymbol{k}_0+\boldsymbol{\psi}_3)$$
$$\times A^*(\boldsymbol{\psi}_1)\,A(\boldsymbol{\psi}_2)\,A(\boldsymbol{\psi}_3)\,e^{i(\boldsymbol{\psi}_2+\boldsymbol{\psi}_3-\boldsymbol{\psi}_1)\cdot\boldsymbol{x}}\,d\boldsymbol{\psi}_1\,d\boldsymbol{\psi}_2\,d\boldsymbol{\psi}_3. \tag{3.8}$$

One can show that the Taylor-series expansion of T, to the lowest order in the spectral width, is given by

$$T(\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0) = T_{\text{I}}+T_{\text{II}} \equiv T_{\text{III}}+T_{\text{IV}}, \tag{3.9a}$$

where

$$T_{\text{I}} = \frac{k_0^3}{32\pi^2\sigma^3}[9\sigma^4-10\sigma^2+9], \quad \sigma = \text{th}\,(k_0 h), \tag{3.9b}$$

$$T_{\text{II}} = -\frac{k_0^3}{32\pi^2\sigma}$$
$$\times\sum_{j=2}^{3}\frac{4c_p^2(\psi_j-\psi_1)^2+4c_p\,c_g(1-\sigma^2)\,(\psi_j-\psi_1)^2+g(1-\sigma^2)^2\,|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|\,\text{th}\,(h|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|)}{g\,|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|\,\text{th}\,(h|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|)-c_g^2(\psi_j-\psi_1)^2} \tag{3.9c}$$

$$T_{\text{III}} = \frac{k_0^3}{32\pi^2\sigma}\left[\frac{g}{\sigma^2}-12+13\sigma^2-2\sigma^4\right], \tag{3.9d}$$

$$T_{\text{IV}} = -\frac{k_0^3}{16\pi^2\sigma}\frac{[2c_p+c_g(1-\sigma^2)^2]^2\,(\psi_2-\psi_1)^2}{g\,\text{th}\,(h|\boldsymbol{\psi}_2-\boldsymbol{\psi}_1|)\,|\boldsymbol{\psi}_2-\boldsymbol{\psi}_1|-c_g^2(\psi_2-\psi_1)^2}, \quad c_p = \frac{\omega}{k_0}. \tag{3.9e}$$

Substituting (3.9d,e) into (3.8) gives

$$i\left(\frac{\partial a}{\partial t}+c_g\frac{\partial a}{\partial x_1}\right)+\frac{c_g}{2k_0}\frac{\partial^2 a}{\partial x_2^2}+\frac{c_g'}{2}\frac{\partial^2 a}{\partial x_1^2} = \alpha_1\,|a|^2\,a+c_g\,\alpha_2\,Ia, \tag{3.10}$$

where

$$\alpha_1 = \frac{g^2k_0^4}{16\omega_0^3}\left(\frac{g}{\sigma^2}-12+13\sigma^2-2\sigma^4\right), \quad \alpha_2 = \frac{k_0^2}{2\omega_0}\left[2\frac{c_p}{c_g}+(1-\sigma^2)\right], \tag{3.11a,b}$$

and

$$I = -\frac{\alpha_2}{4\pi^2}\iint_{-\infty}^{\infty} \frac{(\psi_2-\psi_1)^2 A^*(\boldsymbol{\psi}_1)\, A(\boldsymbol{\psi}_2)\, \mathrm{e}^{\mathrm{i}(\boldsymbol{\psi}_2-\boldsymbol{\psi}_1)\cdot \boldsymbol{x}}}{|\boldsymbol{\psi}_2-\boldsymbol{\psi}_1|\,\mathrm{th}\,(h\,|\boldsymbol{\psi}_2-\boldsymbol{\psi}_1|)-\dfrac{c_g^2}{gh}(\psi_2-\psi_1)^2}\,\mathrm{d}\boldsymbol{\psi}_i\,\mathrm{d}\boldsymbol{\psi}_2. \tag{3.11c}$$

Note that for any finite depth and for $\boldsymbol{\psi}_j-\boldsymbol{\psi}_1 \rightarrow 0$, $j = 2, 3$, the values of T_{II}, T_{IV} and that of the integrand in (3.11*c*) depend on angles θ_j, the 'directions' of approach to the limit, where

$$\cos\theta_j = \lim_{\boldsymbol{\psi}_j\rightarrow\boldsymbol{\psi}_1} \frac{\psi_j-\psi_1}{|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|}.$$

This non-uniqueness disappears for infinitely deep water.

Regarding the integral I, (3.11*c*), one can show that $I \equiv \partial\phi_0/\partial x_1|_{z=0}$, where ϕ_0 is a solution of the following boundary-value problem

$$\frac{\partial^2\phi_0}{\partial x_1^2}+\frac{\partial^2\phi_0}{\partial x_2^2}+\frac{\partial^2\phi_0}{\partial z^2} = 0 \quad (z \leqslant 0), \tag{3.12a}$$

$$\frac{\partial\phi_0}{\partial z}+\frac{c_g^2}{g}\frac{\partial^2\phi_0}{\partial x_1^2} = \frac{c_g\, g\alpha_2}{2\omega_0}\frac{\partial|a|^2}{\partial x_1} \quad (z = 0), \tag{3.12b}$$

$$\frac{\partial\phi_0}{\partial z} = 0 \quad (z = -h). \tag{3.12c}$$

Thus ϕ_0 appears to be the mean-flow potential. The system of equations (3.10) and (3.12) was obtained by Iusim & Stiassnie (1982) using a multiple-scale approach. In the particular case where the water depth is shallow compared with the group length $\mathrm{th}\,(h|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|)$ can be replaced by $h|\boldsymbol{\psi}_j-\boldsymbol{\psi}_1|$ and the set of equations given by Davey & Stewartson (1974) is recovered. For water of infinite depth, Stiassnie (1984) extended the analysis to one order higher in the spectral width, and rederived Dysthe's (1979) set of equations (sometimes called the modified Schrödinger equation); see also Janssen (1983). The fact that the fourth-order (in the wave steepness) modified Schrödinger equation is a particular case of the third-order Zakharov equation (3.1) is less surprising if one realizes that all the fourth-order terms in the modified Schrödinger equation emerge as a result of the narrow-spectral-width assumption, and none of them is of fourth order in the wave amplitude itself.

4. Linear stability of a uniform wavetrain

Following Crawford *et al.* (1981), this section deals with the mathematical formulation of one of the simplest possible non-trivial nonlinear interaction problems and its linearized (short-time) solution. The smallest number of wavetrains required to enable significant nonlinear interaction is three for class I as well as for class II interactions, In what follows, we denote these 3 waves by the subscripts a, b and c. For anything exciting to happen, these 3 waves have to form a nearly resonating 'quartet' for a class I interaction, and a nearly resonating 'quintet' for a class II interaction, see (2.18) and (2.21) respectively.

To form a 'quartet', or a 'quintet', out of three waves, one can 'count' one of the waves $\boldsymbol{k}_a$ twice for class I interactions, and three times for class II interactions.

The governing equations for class I interactions are a discretized form of (3.1):

$$\mathrm{i}\frac{\mathrm{d}B_a}{\mathrm{d}t} = (T_{aaaa}\,|B_a|^2+2T_{abab}\,|B_b|^2+2T_{acac}\,|B_c|^2)\,B_a+2T_{aabc}\,B_a^*\,B_b\,B_c\,\mathrm{e}^{\mathrm{i}\Omega_1 t}, \tag{4.1a}$$

$$\mathrm{i}\frac{\mathrm{d}B_b}{\mathrm{d}t} = (2T_{baba}|B_a|^2+T_{bbbb}|B_b|^2+2T_{bcbc}|B_c|^2)\,B_b+T_{bcaa}\,B_c^*\,B_a^2\,\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{I}}t}, \qquad (4.1\,b)$$

$$\mathrm{i}\frac{\mathrm{d}B_c}{\mathrm{d}t} = (2T_{caca}|B_a|^2+2T_{cbcb}|B_b|^2+T_{cccc}|B_c|^2)\,B_c+T_{cbaa}\,B_b^*\,B_a^2\,\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{I}}t} \qquad (4.1\,c)$$

where $\Omega_{\mathrm{I}} = 2\omega_a-\omega_b-\omega_c$.

For class II 3-wave problems, which do not satisfy (2.18), (2.24) similarly gives

$$\mathrm{i}\frac{\mathrm{d}B_a}{\mathrm{d}t} = (T_{aaaa}|B_a|^2+2T_{abab}|B_b|^2+2T_{acac}|B_c|^2)\,B_a+2U^{(3)}_{aaabc}\,(B_a^*)^2B_b\,\mathrm{e}^{\mathrm{i}\Omega_{\mathrm{II}}t}, \quad (4.2\,a)$$

$$\mathrm{i}\frac{\mathrm{d}B_b}{\mathrm{d}t} = (2T_{baba}|B_a|^2+T_{bbbb}|B_b|^2+2T_{bcbc}|B_c|^2)\,B_b+U^{(2)}_{bcaaa}\,B_c^*\,B_a^3\,\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{II}}t}, \qquad (4.2\,b)$$

$$\mathrm{i}\frac{\mathrm{d}B_c}{\mathrm{d}t} = (2T_{caca}|B_a|^2+2T_{cbcb}|B_b|^2+T_{cccc}|B_c|^2)\,B_b+U^{(2)}_{cbaaa}\,B_b^*\,B_a^3\,\mathrm{e}^{-\mathrm{i}\Omega_{\mathrm{II}}t}, \qquad (4.2\,c)$$

where $\Omega_{\mathrm{II}} = 3\omega_a-\omega_b-\omega_c$ and $U^{(3)}_{aaabc}$ is assumed to be symmetric with respect to b and c.

To complete the mathematical formulation of either of the above systems of equations, (4.1) or (4.2), one has to specify the following initial conditions:

$$B_a(0) = b_a, \quad B_b(0) = b_b, \quad B_c(0) = b_c,$$

where the relation between B_j and the actual physical amplitude a_j is

$$a_j = \frac{1}{\pi}\left(\frac{\omega_j}{2g}\right)^{\frac{1}{2}}|B_j|.$$

One can assume that the initial amplitude of one of the waves, which is called the 'carrier' and denoted by the subscript a, is much larger than the amplitudes of the other two waves, denoted by b and c, to be called the 'disturbances': $|b_b|$, $|b_c| \ll |b_a|$. Only linear terms in the disturbances B_b, B_c are retained, so that the carrier wave remains unaffected in this short-time analysis, and is given by $B_a = b_a\,\mathrm{e}^{-\mathrm{i}T_{aaaa}|b_a|^2t}$; b_a is assumed to be real without loss of generality.

4.1. *Class I instabilities*

The wavenumbers of the carrier and the disturbances are

$$\boldsymbol{k}_a = k_0(1,0), \quad \boldsymbol{k}_b = k_0(1+p,q), \quad \boldsymbol{k}_c = k_0(1-p;-q), \qquad (4.3\,a,b,c)$$

so that (2.18a) is satisfied identically. The linearized version of (4.1b, c) is

$$\mathrm{i}\frac{\mathrm{d}B_b}{\mathrm{d}t} = 2T_{baba}\,b_a^2\,B_b+T_{bcaa}\,B_c^*\,b_a^2\,\mathrm{e}^{-\mathrm{i}\tilde{\Omega}_{\mathrm{I}}t}, \qquad (4.4\,a)$$

$$\mathrm{i}\frac{\mathrm{d}B_c}{\mathrm{d}t} = 2T_{caca}\,b_a^2\,B_c+T_{cbaa}\,B_b^*\,b_a^2\,\mathrm{e}^{-\mathrm{i}\tilde{\Omega}_{\mathrm{I}}t}, \qquad (4.4\,b)$$

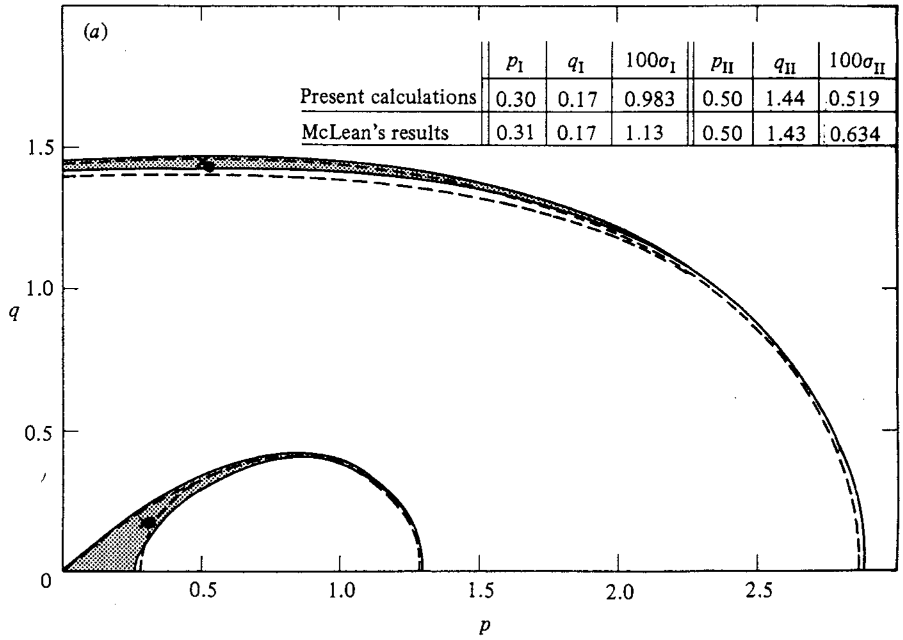
where $\tilde{\Omega}_{\mathrm{I}} = \omega_{\mathrm{I}}+2T_{aaaa}\,b_a^2$.

Assuming a solution of the form

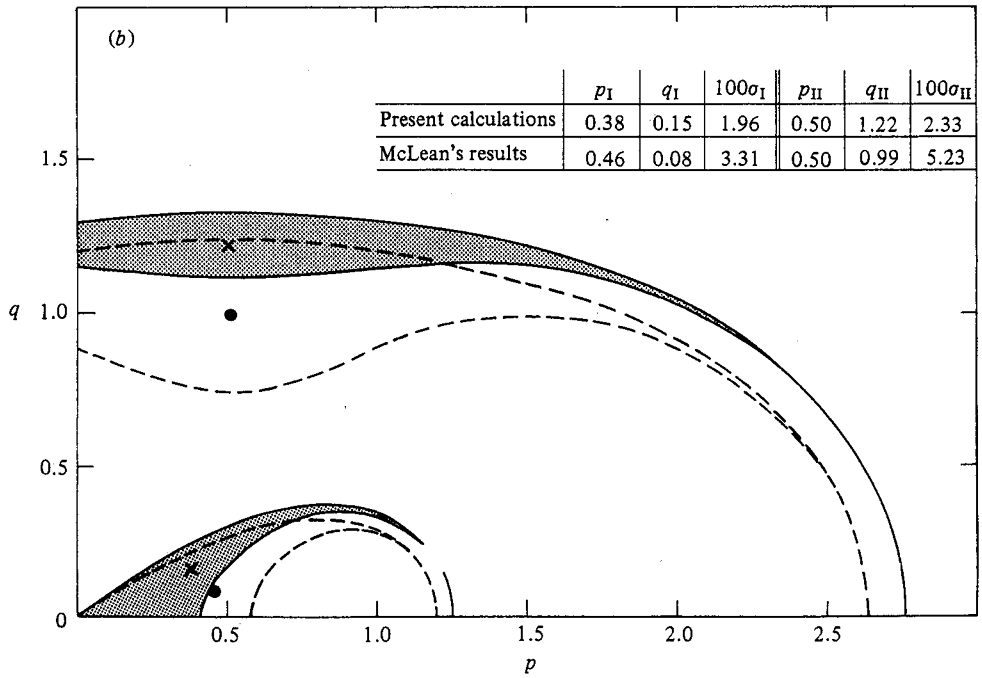
$$B_b = b_b\,\mathrm{e}^{-\mathrm{i}(0.5\tilde{\Omega}_{\mathrm{I}}+\delta_{\mathrm{I}})\,t}, \quad B_c = b_c\,\mathrm{e}^{-\mathrm{i}(0.5\tilde{\Omega}_{\mathrm{I}}-\delta_{\mathrm{I}})\,t},$$

one can show that δ_{I} must be given by

$$\delta_{\mathrm{I}} = (T_{baba}-T_{caca})\,b_a^2 \pm D_{\mathrm{I}}^{\frac{1}{2}}, \qquad (4.5\,a)$$



	p_{I}	q_{I}	$100\sigma_{\mathrm{I}}$	p_{II}	q_{II}	$100\sigma_{\mathrm{II}}$
Present calculations	0.30	0.17	0.983	0.50	1.44	0.519
McLean's results	0.31	0.17	1.13	0.50	1.43	0.634



	p_{I}	q_{I}	$100\sigma_{\mathrm{I}}$	p_{II}	q_{II}	$100\sigma_{\mathrm{II}}$
Present calculations	0.38	0.15	1.96	0.50	1.22	2.33
McLean's results	0.46	0.08	3.31	0.50	0.99	5.23

FIGURE 1 (*a*, *b*). For caption see facing page.

where

$$D_{\mathrm{I}} = [0.5\tilde{\Omega}_{\mathrm{I}} - (T_{baba} + T_{caca})\, b_a^2]^2 - T_{bcaa}\, T_{cbaa}\, b_a^4. \tag{4.5b}$$

Positive values of $D_{\mathrm{I}}(p,q)$ correspond to stability regions in the (p,q)-plane and *vice versa*. The curves $D_{\mathrm{I}} = 0$ form the stability boundaries, and the point where D_{I} attains its minimum is called the most-unstable mode. The value of $\sigma_{\mathrm{I}} = (-D_{\mathrm{I}}/gk_0)^{\frac{1}{2}}$ for the most-unstable mode is called the maximum growth rate.

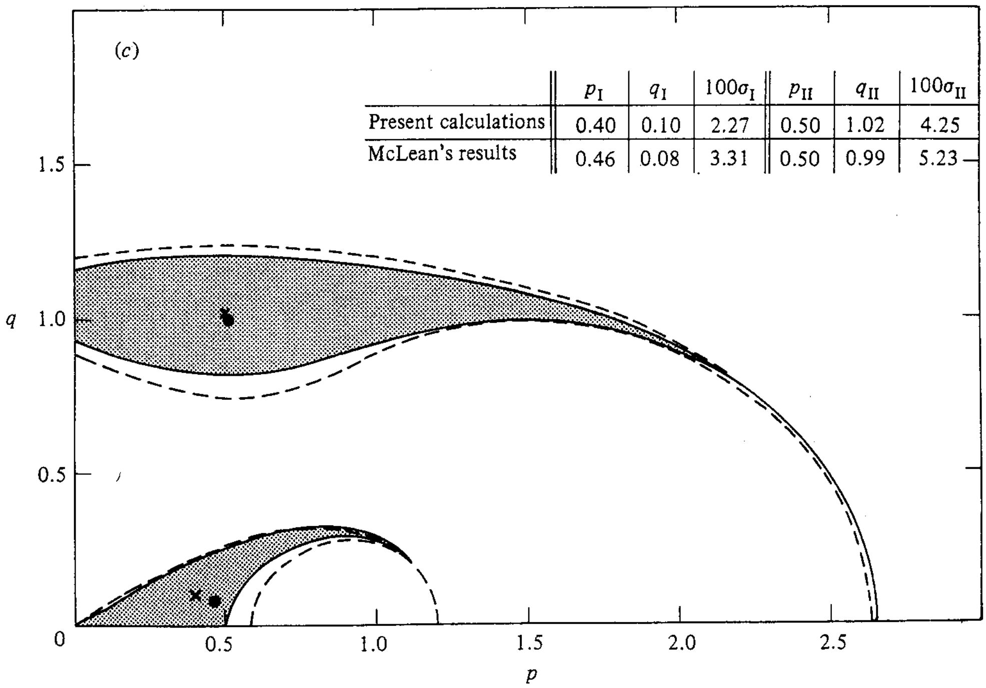


FIGURE 1. Bands of instability for $k_0 h = 2$. The instability boundaries are given by the solid lines and the points of maximum growth rate are labelled by ×. McLean's results are marked by the dashed lines and by ●. (*a*) $k_0 a_0 = 0.195$, $(ka)_M = 0.2$; (*b*) 0.326, 0.35; (*c*) 0.41, 0.35.

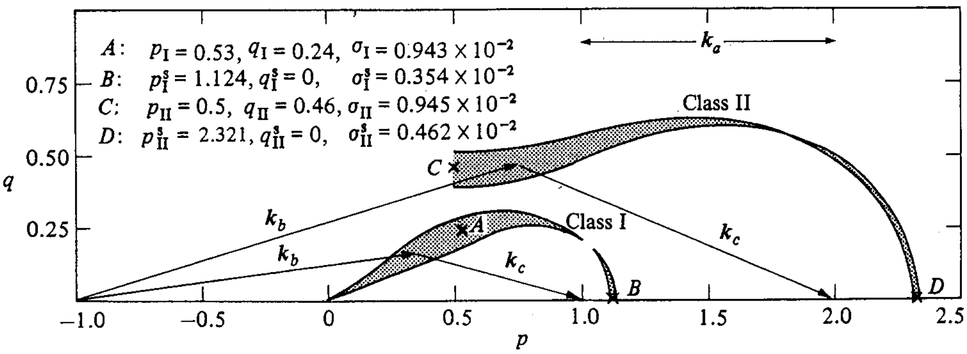


FIGURE 2. Bands of instability for $k_0 h = 0.35$, $k_0 a_0 = 0.04$ and notation.

4.2. *Class II instabilities*

For this case the carrier wavenumber $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$ are still given by (4.3*a*, *b*), but

$$\boldsymbol{k}_c = k_0(2-p, -q), \tag{4.6}$$

so that (2.21*a*) is now satisfied identically.

The linearized short-time version of (4.2*b*, *c*) is

$$\mathrm{i}\frac{\mathrm{d}B_b}{\mathrm{d}t} = 2T_{baba}\, b_a^2\, B_b + U^{(2)}_{bcaaa}\, B_c^*\, b_a^3\, \mathrm{e}^{-\mathrm{i}\Omega_{II} t}, \tag{4.7a}$$

$$\mathrm{i}\frac{\mathrm{d}B_c}{\mathrm{d}t} = 2T_{caca}\, b_a^2\, B_c + U^{(2)}_{cbaaa}\, B_b^*\, b_a^3\, \mathrm{e}^{-\mathrm{i}\tilde{\Omega}_{II} t}, \tag{4.7b}$$

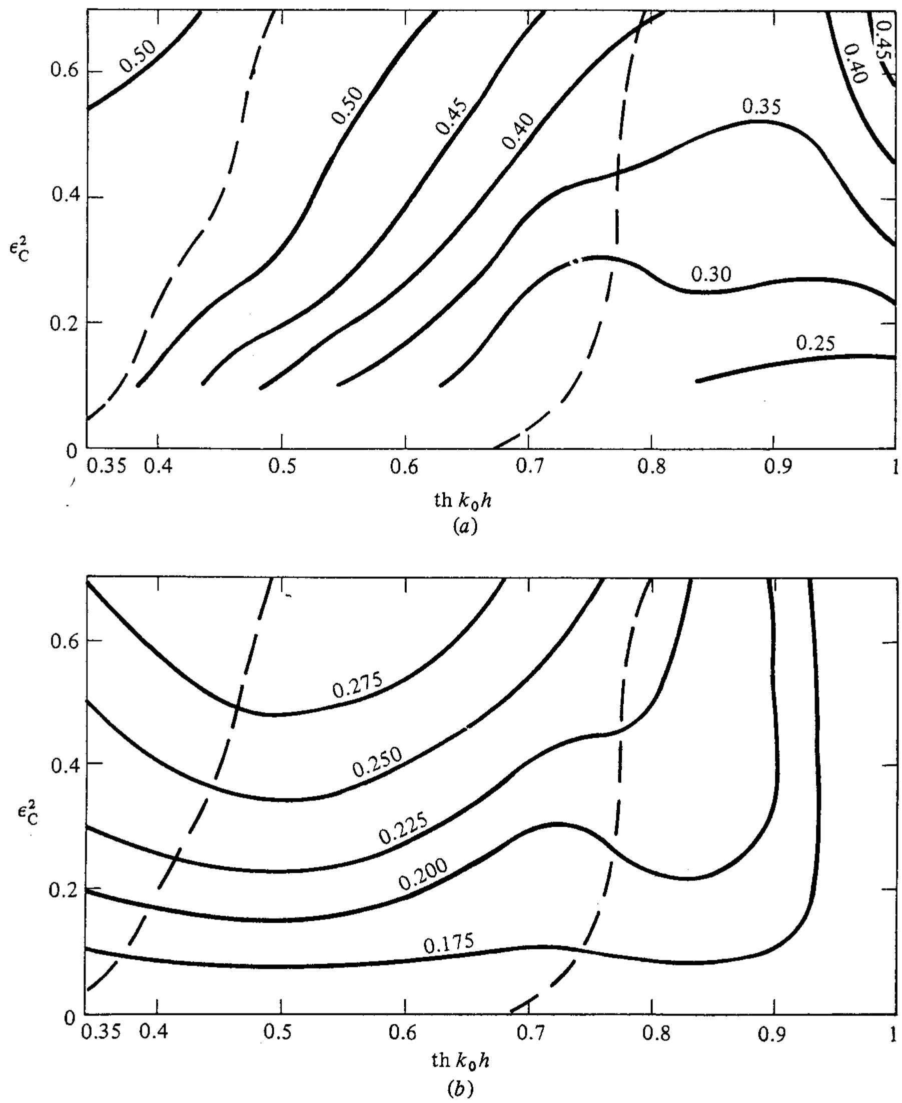


FIGURE 3(*a*, *b*). For caption see facing page.

where $\tilde{\Omega}_{\rm II} = \Omega_{\rm II} + 3T_{aaaa}\, b_a^2$. Assuming again a solution of the form

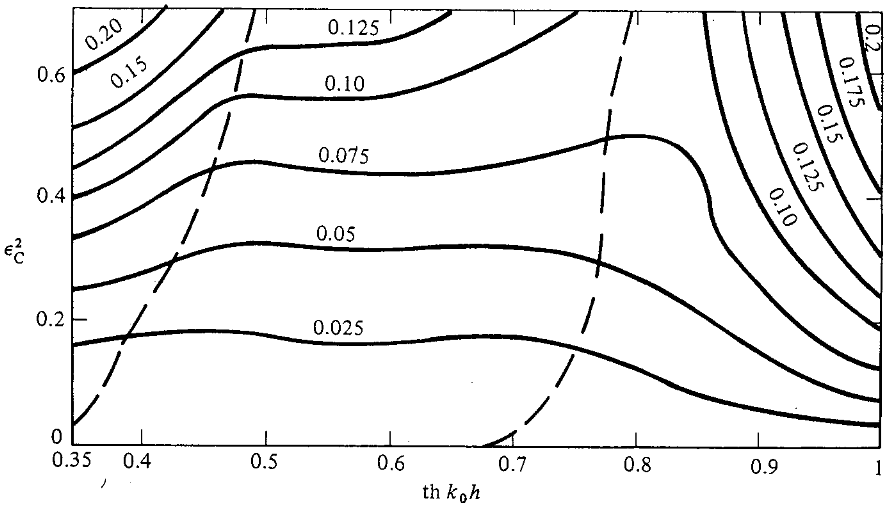
$$B_b = b_b\, e^{-i(0.5\tilde{\Omega}_{\rm II}+\delta_{\rm II})\,t}, \quad B_c = b_c\, e^{-i(0.5\tilde{\Omega}_{\rm II}-\delta_{\rm II})\,t},$$

one finds that

$$\delta_{\rm II} = (T_{baba} - T_{caca})\, b_a^2 \pm D_{\rm II}^{\frac{1}{2}}, \tag{4.8a}$$

$$D_{\rm II} = [0.5\tilde{\Omega}_{\rm II} - (T_{baba}+T_{caca})\, b_a^2]^2 - U^{(2)}_{bcaaa}\, U^{(2)}_{cbaaa}\, b_a^6. \tag{4.8b}$$

The stability boundary and maximum growth rate for class II interactions are obtained from (4.8*b*).



(*c*)

FIGURE 3. Summary of results for class I instabilities: (*a*) isolines of p_I (for $\epsilon_C^2 = 0$, $p_I = 0$); (*b*) isolines of q_I (for either $\epsilon_C^2 = 0$ or th $k_0 h = 1$, $q_I = 0$); (*c*) isolines of $10\sigma_I^M$, the maximum growth rate (for $\epsilon_C^2 = 0$, $\sigma_I^M = 0$).

4.3. *Results*

In figure 1 we show the class I and class II instability regions (as shaded zones) for $k_0 h = 2$. The solid lines represent the calculated results and the dashed curves are those of McLean (1982*b*, figures 2*b*, *c*). In figure 1 (*a*), $k_0 a_0 = 0.195$ (where a_0 is the first-order amplitude of the carrier in Stokes' expansion), which is equivalent to $(ka)_M = 0.2$ (the subscript M stands for McLean). As a conversion formula we used the following expression:

$$(ka)_M = k_0 a_0 + \frac{24ch^6(k_0 h)+3}{64sh^6(k_0 h)}(k_0 a_0)^3 + O(k_0 a_0)^5, \tag{4.9}$$

given by Skjelbreia & Hendrickson (1961). In figure 1 (*b*) $k_0 a_0 = 0.326$ (corresponding to $(ka)_M = 0.35$).

The locations of the maximum growth rates (p_I, q_I), (p_{II}, q_{II}) for class I and class II instabilities respectively are marked by × for our results and by a dot for McLean's results, and their numerical values, as well as those of the maximum growth rates σ_I, σ_{II}, are given in the figures. The overall agreement in figure 1 (*a*) is quite satisfactory. The actual amplitude in this figure is 47 % of the theoretical maximum (Cokelet 1977). For smaller steepnesses the agreement becomes even better. On the other hand, for very steep waves (in figure 1 (*b*) the actual amplitude is 82 % of the theoretical maximum amplitude) the agreement is less impressive. Nevertheless, a somewhat better agreement is obtained if we compare McLean's results for $(ka)_M = 0.35$ with the artificially amplified value $a_0 k_0 = 0.41$ (see figure 1 *c*). A similar trend in the degree of agreement between the results of the present model and those of McLean was obtained for several other water depths. Generally speaking, the present model gives good quantitative results for amplitudes which are less than about one-half of the theoretical maximum. For very steep waves the present model loses its quantitative validity, but still predicts the general qualitative features.

Figure 2, which is quite typical, is used to demonstrate some general features as well as clarify some of the terminology which is used later.

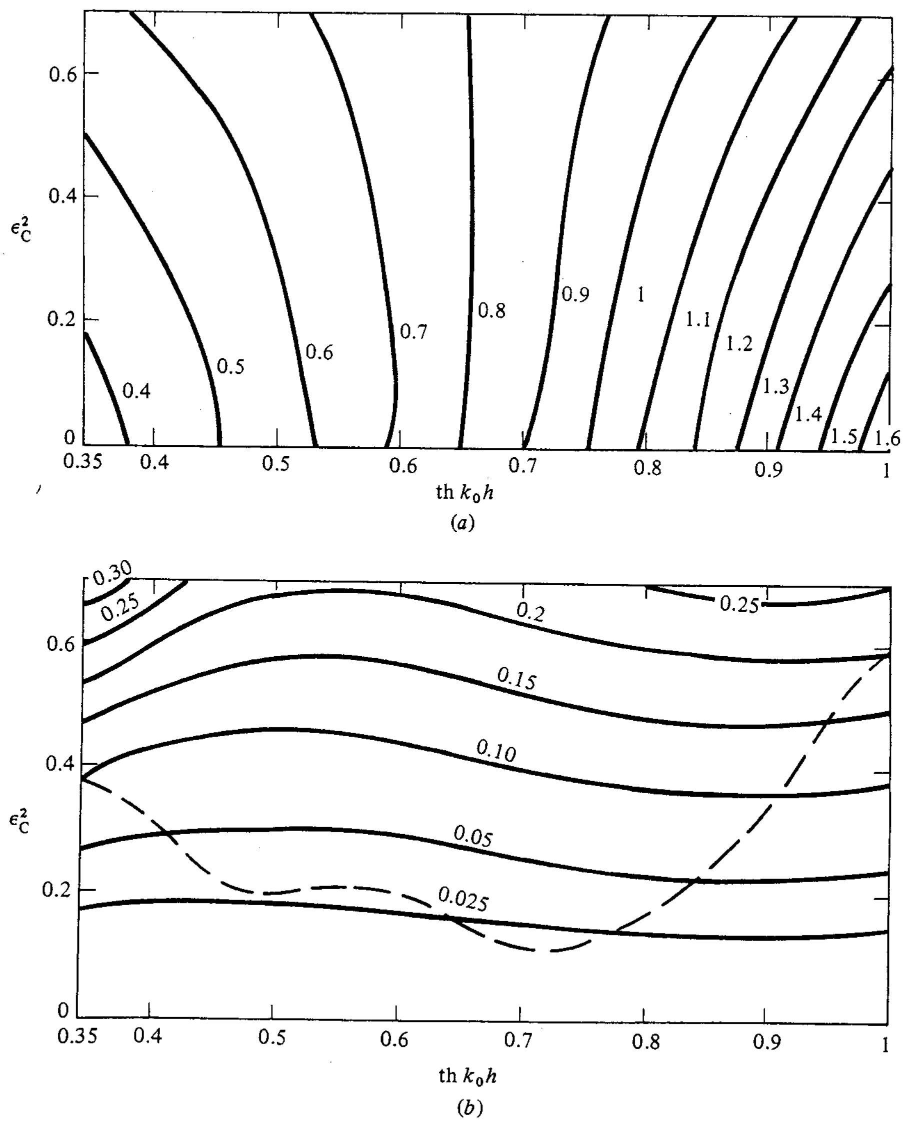


FIGURE 4. Summary of results for class II instabilities; (a) isolines of q_{II}; (b) isolines of $10\sigma_{II}$.

One can see that a certain similarity exists between class I and class II instability regions. Both can be regarded as consisting of two domains: a wider band at lower values of p and usually a much narrower region at higher values of p. The first region will be referred to as the main region, and the other as the secondary instability region. The difference between class I and class II instability regions is that for class I the two domains are usually disconnected while in the case of class II they are bound by a line of infinitesimal thickness. The secondary regions sometimes disappear completely, and for class I, the instability region in these cases terminates at some $q > 0$ (compare with Crawford *et al.* (1981) for infinite water depth).

The disconnection between the main and secondary regions for class I, as well as the disappearance of the secondary region in some cases, are probably results of the order of the present perturbation expansion, since they are not observed in McLean

(1982*a*, *b*), who used the full equations. However, the present results seem to indicate that the instabilities for these cases are of higher order.

Figure 2 shows the three wavenumber vectors $\boldsymbol{k}_a$, $\boldsymbol{k}_b$, $\boldsymbol{k}_c$, as well as the location of four points of local maximum growth rates:

A, class I point $(p_{\mathrm{I}}, q_{\mathrm{I}})$ with local maximum growth rate σ_{I};

B, the secondary class I point $(p_{\mathrm{I}}^{\mathrm{s}}, q_{\mathrm{I}}^{\mathrm{s}})$ with local maximum growth rate $\sigma_{\mathrm{I}}^{\mathrm{s}}$;

C, class II point $(p_{\mathrm{II}}, q_{\mathrm{II}})$ with local maximum growth rate σ_{II};

D, the secondary class II point $(p_{\mathrm{II}}^{\mathrm{s}}, q_{\mathrm{II}}^{\mathrm{s}})$ with local maximum growth rate $\sigma_{\mathrm{II}}^{\mathrm{s}}$.

For the particular data of figure 2 ($k_0 h = 0.35$, $k_0 a_0 = 0.04$) $\sigma_{\mathrm{II}} > \sigma_{\mathrm{I}} > \sigma_{\mathrm{II}}^{\mathrm{s}} > \sigma_{\mathrm{I}}^{\mathrm{s}}$. These inequalities are by no means general, as will be shown in the sequel. Nevertheless, for most cases $\sigma^{\mathrm{II}} > \sigma_{\mathrm{s}}^{\mathrm{II}}$.

Figure 3 is a summary of the results for class I instabilities. Figures 3 (*a*, *b*) give the values of p_{I} and q_{I} respectively as functions of the water depth and wave steepness. The depth is expressed by $\operatorname{th} k_0 h$ (the range covered is $0.357 \leqslant k_0 h \leqslant \infty$), and the wave steepness by Cokelet's (1977) ϵ^2, denoted here by ϵ_{C}^2 (the range $0 \leqslant \epsilon_{\mathrm{C}}^2 < 0{,}7$ is covered).

The isolines in figures 3 and 4 were drawn using interpolation and are based on about forty computed data points, almost equally distributed over the figure domain. Figure 3 (*c*) is a plot of $\sigma_{\mathrm{I}}^{\mathrm{M}} = \max(\sigma_{\mathrm{I}}, \sigma_{\mathrm{I}}^{\mathrm{s}})$ isolines. Note that for the region confined by the broken lines $\sigma_{\mathrm{I}}^{\mathrm{s}} > \sigma_{\mathrm{I}}$ (sometimes by a factor of three), whereas the opposite is true in the outside region. For the case where $\sigma_{\mathrm{I}}^{\mathrm{s}} > \sigma_{\mathrm{I}}$, $p_{\mathrm{I}}^{\mathrm{s}}$ is in the range 1.05–1.30 and $q_{\mathrm{I}}^{\mathrm{s}} = 0$, which implies that the most-unstable mode is two-dimensional.

The results for class II are given in figure 4. Note that for this case p_{II} is always 0.5. Figure 4 (*a*) gives the values of q_{II}, and σ_{II} is shown in figure 4 (*b*). For the domain above the dashed line in figure 4 (*b*), $\sigma_{\mathrm{II}} > \sigma_{\mathrm{I}}^{\mathrm{M}}$, which indicates that for this region class II instabilities may become dominant.

The question whether the disturbances related to the highest value of σ will dominate the physical process remains open, and awaits additional evidence. The authors hope that their current study of the long-time evolution of class I and class II instabilities will throw some light on this and on other relevant aspects of these important processes.

Appendix

The kernels in (2.9) are

$$S^{(1)}_{0,1,2,3} = \frac{\omega_0^2}{g} J^{(1)}_{0,1,2,3} - \frac{\omega_1^2}{2g}\{|\boldsymbol{k}_1|^2 - \tfrac{1}{2}(|\boldsymbol{k}-\boldsymbol{k}_2|^2 + |\boldsymbol{k}-\boldsymbol{k}_3|^2 + |\boldsymbol{k}_1+\boldsymbol{k}_2|^2 + |\boldsymbol{k}_1+\boldsymbol{k}_3|^2)\}, \tag{A 1a}$$

$$S^{(2)}_{0,1,2,3,4} = -\tfrac{1}{6}|\boldsymbol{k}_1|^4 + \frac{\omega_0^2}{g} J^{(2)}_{0,1,2,3,4} + |\boldsymbol{k}-\boldsymbol{k}_2|^2 J^{(1)}_{0-2,1,3,4} + \frac{|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}{2}\left(\frac{\omega_1\,\omega_{1+2}}{g}\right)^2, \tag{A 1b}$$

where

$$J^{(1)}_{0,1,2,3} = \frac{|\boldsymbol{k}_1|}{4}\left\{2\,|\boldsymbol{k}_1| - \frac{\operatorname{th}|\boldsymbol{k}_1|\,h}{g}(\omega_{0-2}^2 + \omega_{0-3}^2 + \omega_{1+2}^2 + \omega_{1+3}^2)\right\}, \tag{A 1c}$$

$$J^{(2)}_{0,1,2,3,4} = \frac{|\boldsymbol{k}_1|^3}{6}\operatorname{th}|\boldsymbol{k}_1|\,h - \frac{\omega_1^2}{2g}|\boldsymbol{k}_1+\boldsymbol{k}_2|^2 - \frac{\omega_{0-2}^2}{g} J^{(1)}_{0-2,1,3,4}. \tag{A 1d}$$

The interaction coefficients in (2.11) as as follows:

Second order:

$$V^{(1)}_{0,1,2} = -2V_{-0,1,2} - V_{1,2,0}, \quad \text{(A 2a)}$$

$$V^{(2)}_{0,1,2} = 2(V_{0,1,2} - V_{-0,2,1} - V_{-1,2,0}), \quad \text{(A 2b)}$$

$$V^{(3)}_{0,1,2} = 2V_{0,1,2} + V_{1,2,0}, \quad \text{(A 2c)}$$

where

$$V_{0,1,2} = \frac{1}{8\pi}\left(\frac{g\omega_2}{2\omega_0\omega_1}\right)^{\frac{1}{2}}\left[\boldsymbol{k}\cdot\boldsymbol{k}_1 + \left(\frac{\omega_0\omega_1}{g}\right)^2\right]. \quad \text{(A 2d)}$$

Third order:

$$W^{(1)}_{0,1,2,3} = W_{1,2,-0,3} - W_{-0,1,2,3}, \quad \text{(A 3a)}$$

$$W^{(2)}_{0,1,2,3} = W_{-0,-1,2,3} + W_{2,3,-0,-1} - W_{2,-1,-0,3} - W_{-0,2,-1,3} - W_{-0,3,2,-1} - W_{3,-1,2,-0}, \quad \text{(A 3b)}$$

$$W^{(3)}_{0,1,2,3} = 2W_{-0,-1,-2,3} - W_{-0,3,-1,-2} + W_{-1,-2,-0,3} - 2W_{-1,3,-0,-2}, \quad \text{(A 3c)}$$

$$W^{(4)}_{0,1,2,3} = W_{0,1,2,3} + W_{1,2,0,3}, \quad \text{(A 3d)}$$

where

$$W_{0,1,2,3} = \frac{1}{64\pi^2}\left(\frac{\omega_2\omega_3}{\omega_0\omega_1}\right)^{\frac{1}{2}}|\boldsymbol{k}||\boldsymbol{k}_1|\Big\{2|\boldsymbol{k}|\operatorname{th}|\boldsymbol{k}_1|h + 2|\boldsymbol{k}_1|\operatorname{th}|\boldsymbol{k}|h - \frac{1}{g}\operatorname{th}|\boldsymbol{k}|h\operatorname{th}|\boldsymbol{k}_1|h\,[\omega^2_{0+2}+\omega^2_{0+3}+\omega^2_{1+2}+\omega^2_{1+3}]\Big\}. \quad \text{(A 3e)}$$

Fourth order:

$$X^{(1)}_{0,1,2,3,4} = \alpha_{0,1,2,3,4} + \beta_{0,1,2,3,4}, \quad \text{(A 4a)}$$

$$X^{(2)}_{0,1,2,3,4} = \alpha_{0,4,2,3,-1} + \beta_{0,4,2,3,-1} + \alpha_{0,3,2,-1,4} + \beta_{0,3,2,-1,4} + \alpha_{0,2,-1,3,4} - \beta_{0,2,-1,3,4} - \alpha_{0,-1,2,3,4} - \beta_{0,-1,2,3,4}, \quad \text{(A 4b)}$$

$$X^{(3)}_{0,1,2,3,4} = \alpha_{0,3,4,-1,-2} + \beta_{0,3,4,-1,-2} - \alpha_{0,-1,3,-2,4} - \beta_{0,-1,3,-2,4} - \alpha_{0,-1,4,3,-2} - \beta_{0,-1,4,3,-2} + \alpha_{0,3,-2,-1,4} - \beta_{0,3,-2,-1,4} + \alpha_{0,4,-2,3,-1} - \beta_{0,4,-2,3,-1} - \alpha_{0,-1,-2,3,4} + \beta_{0,-1,-2,3,4}, \quad \text{(A 4c)}$$

$$X^{(4)}_{0,1,2,3,4} = -\alpha_{0,-1,4,-2,-3} - \beta_{0,-1,4,-2,3} + \alpha_{0,4,-2,-3,-1} - \beta_{0,4,-2,-3,-1} - \alpha_{0,-1,-2,-3,4} + \beta_{0,-1,-2,-3,4} - \alpha_{0,-1,-2,4,-3} + \beta_{0,-1,-2,4,-3}, \quad \text{(A 4d)}$$

$$X^{(5)}_{0,1,2,3,4} = -\alpha_{0,-1,-2,-3,-4} + \beta_{0,-1,-2,-3,-4}, \quad \text{(A 4e)}$$

where

$$\alpha_{0,1,2,3,4} = -\frac{1}{32\pi^3}\left(\frac{\omega_2\omega_3\omega_4}{2g\omega_0\omega_1}\right)^{\frac{1}{2}}\left\{S^{(2)}_{0,1,2,3,4} - |\boldsymbol{k}_1|(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\left(\frac{\omega^2_{1+4}}{g} - |\boldsymbol{k}_1|\right)\right\}, \quad \text{(A 4f)}$$

$$\beta_{0,1,2,3,4} = -\frac{1}{32\pi^3}\left(\frac{\omega_0\omega_3\omega_4}{2g\omega_1\omega_2}\right)^{\frac{1}{2}}\left\{\frac{\omega_1^2}{g}S^{(1)}_{0-1,2,3,4} + \frac{(\boldsymbol{k}_2\cdot\boldsymbol{k}_4)}{2}\left(\frac{\omega_1\omega_2}{g}\right)^2 - \frac{|\boldsymbol{k}_1||\boldsymbol{k}_2|}{2}\left[\frac{\omega^2_{1+4}}{g}\operatorname{th}|\boldsymbol{k}_1|h - |\boldsymbol{k}_1|\right]\left[\frac{\omega^2_{2+3}}{g}\operatorname{th}|\boldsymbol{k}_2|h - |\boldsymbol{k}_2|\right]\right\}. \quad \text{(A 4g)}$$

The kernels of (2.15) are

$$\tilde{T}^{(1)}_{0,1,2,3} = W^{(1)}_{0,1,2,3} - \frac{V^{(1)}_{0,1,2+3}\,V^{(1)}_{2+3,3,2}}{\omega_{2+3}-\omega_2-\omega_3} - \frac{V^{(1)}_{0,1+3,2}\,V^{(1)}_{1+3,3,1}}{\omega_{1+3}-\omega_1-\omega_3} - \frac{V^{(2)}_{0,-1-3,2}\,V^{(3)}_{-1-3,3,1}}{\omega_{1+3}+\omega_1+\omega_3}, \tag{A 5a}$$

$$\begin{aligned}\tilde{T}^{(2)}_{0,1,2,3} = W^{(2)}_{0,1,2,3} &- \frac{V^{(1)}_{0,2,3-1}\,V^{(2)}_{3-1,1,3}}{\omega_{3-1}+\omega_1-\omega_3} - \frac{V^{(1)}_{0,3-1,2}\,V^{(2)}_{3-1,1,3}}{\omega_{3-1}+\omega_1-\omega_3} - \frac{V^{(2)}_{0,2-0,2}\,V^{(2)}_{1-3,3,1}}{\omega_{1-3}+\omega_3-\omega_1} \\ &- \frac{V^{(3)}_{0,-0-1,1}\,V^{(3)}_{-2-3,3,2}}{\omega_{2+3}+\omega_2+\omega_3} - \frac{V^{(3)}_{0,1,-0-1}\,V^{(3)}_{-2-3,3,2}}{\omega_{2+3}+\omega_2+\omega_3} - \frac{V^{(1)}_{2+3,2,3}\,V^{(2)}_{0,1,0+1}}{\omega_{2+3}-\omega_2-\omega_3},\end{aligned} \tag{A 5b}$$

$$T_{0,1,2,3} = 0.5(\tilde{T}^{(2)}_{0,1,2,3}+\tilde{T}^{(2)}_{0,1,3,2}), \tag{A 5c}$$

$$\begin{aligned}\tilde{T}^{(3)}_{0,1,2,3} = W^{(3)}_{0,1,2,3} &- \frac{V^{(2)}_{0,1+2,3}\,V^{(1)}_{1+2,2,1}}{\omega_{1+2}-\omega_1-\omega_2} - \frac{V^{(2)}_{0,1,-2+3}\,V^{(2)}_{-2+3,2,3}}{\omega_{2-3}+\omega_2-\omega_3} - \frac{V^{(3)}_{0,1,2-3}\,V^{(2)}_{2-3,3,2}}{\omega_{2-3}+\omega_3-\omega_2} \\ &- \frac{V^{(3)}_{0,1-3,2}\,V^{(2)}_{1-3,3,1}}{\omega_{1-3}+\omega_3-\omega_1} - \frac{V^{(1)}_{0,3,-1-2}\,V^{(3)}_{-1-2,1,2}}{\omega_{1+2}+\omega_1+\omega_2} - \frac{V^{(1)}_{0,-1-2,3}\,V^{(3)}_{-1-2,2,1}}{\omega_{1+2}+\omega_1+\omega_2},\end{aligned} \tag{A 5d}$$

$$\tilde{T}^{(4)}_{0,1,2,3} = W^{(4)}_{0,1,2,3} - \frac{V^{(3)}_{0,1,2+3}\,V^{(1)}_{2+3,3,2}}{\omega_{2+3}-\omega_2-\omega_3} - \frac{V^{(3)}_{0,1+3,2}\,V^{(1)}_{1+3,3,1}}{\omega_{1+3}-\omega_1-\omega_3} - \frac{V^{(2)}_{0,1,-2-3}\,V^{(3)}_{-2-3,3,2}}{\omega_{2+3}+\omega_2+\omega_3}. \tag{A 5e}$$

The kernels of (2.20) are

$$\begin{aligned}\tilde{U}^{(2)}_{0,1,2,3,4} &= X^{(2)}_{0,1,2,3,4} + \frac{V^{(1)}_{0,3+4,2-1}\,V^{(1)}_{3+4,3,4}\,V^{(2)}_{2-1,1,2}}{(\omega_{3+4}-\omega_3-\omega_4)(\omega_{2-1}+\omega_1-\omega_2)} + \frac{V^{(1)}_{0,4-1,2+3}\,V^{(2)}_{4-1,1,4}\,V^{(1)}_{2+3,2,3}}{(\omega_{4-1}+\omega_1-\omega_4)(\omega_{2+3}-\omega_2-\omega_3)} \\ &+ \frac{V^{(2)}_{0,1-3,2+4}\,V^{(2)}_{1-3,3,1}\,V^{(1)}_{2+4,2,4}}{(\omega_{1-3}+\omega_3-\omega_1)(\omega_{2+4}-\omega_2-\omega_4)} + \frac{V^{(2)}_{0,-3-4,2-1}\,V^{(3)}_{-3-4,3,4}\,V^{(2)}_{2-1,1,2}}{(\omega_{3+4}+\omega_3+\omega_4)(\omega_{2-1}+\omega_1-\omega_2)} \\ &+ \frac{V^{(3)}_{0,1-3,-2-4}\,V^{(2)}_{1-3,3,1}\,V^{(3)}_{-2-4,4,2}}{(\omega_{1-3}+\omega_3-\omega_1)(\omega_{2+4}+\omega_2+\omega_4)} + \frac{V^{(3)}_{0,-3-4,1-2}\,V^{(3)}_{-3-4,3,4}\,V^{(2)}_{1-2,2,1}}{(\omega_{3+4}+\omega_3+\omega_4)(\omega_{1-2}+\omega_2-\omega_1)} \\ &- \frac{V^{(1)}_{0,2,3+4-1}\,\tilde{T}^{(2)}_{3+4-1,1,4,3}}{\omega_{3+4-1}+\omega_1-\omega_3-\omega_4} - \frac{V^{(1)}_{0,3+4-1,2}\,\tilde{T}^{(2)}_{3+4-1,1,4,3}}{\omega_{3+4-1}+\omega_1-\omega_3-\omega_4} - \frac{V^{(2)}_{0,1,2+3+4}\,\tilde{T}^{(1)}_{2+3+4,3,4,2}}{\omega_{2+3+4}-\omega_2-\omega_3-\omega_4} \\ &- \frac{V^{(2)}_{0,1-3-4,2}\,\tilde{T}^{(3)}_{1-3-4,3,4,1}}{\omega_{1-3-4}+\omega_3+\omega_4-\omega_1} - \frac{V^{(3)}_{0,1,-2-3-4}\,\tilde{T}^{(4)}_{-2-3-4,3,4,2}}{\omega_{2+3+4}+\omega_2+\omega_3+\omega_4} - \frac{V^{(3)}_{0,-2-3-4,1}\,\tilde{T}^{(4)}_{-2-3-4,3,4,2}}{\omega_{2+3+4}+\omega_2+\omega_3+\omega_4} \\ &- \frac{V^{(2)}_{4-1,1,4}\,W^{(1)}_{0,4-1,2,3}}{\omega_{4-1}+\omega_1-\omega_4} - \frac{V^{(2)}_{4-1,1,4}\,W^{(1)}_{0,2,4-1,3}}{\omega_{4-1}+\omega_1-\omega_4} - \frac{V^{(2)}_{3-1,1,3}\,W^{(1)}_{0,4,2,3-1}}{\omega_{3-1}+\omega_1-\omega_3} \\ &- \frac{V^{(2)}_{1-4,4,1}\,W^{(2)}_{0,1-4,2,3}}{\omega_{1-4}+\omega_4-\omega_1} - \frac{V^{(1)}_{2+4,4,2}\,W^{(2)}_{0,1,2+4,3}}{\omega_{2+4}-\omega_2-\omega_4} - \frac{V^{(1)}_{3+4,4,3}\,W^{(2)}_{0,1,2,3+4}}{\omega_{3+4}-\omega_3-\omega_4} \\ &- \frac{V^{(3)}_{-2-4,4,2}\,W^{(3)}_{0,-2-4,1,3}}{\omega_{2+4}+\omega_2+\omega_4} - \frac{V^{(3)}_{-2-4,4,2}\,W^{(3)}_{0,1,-2-4,3}}{\omega_{2+4}+\omega_2+\omega_4},\end{aligned} \tag{A 6a}$$

$$\begin{aligned}\tilde{U}^{(3)}_{0,1,2,3,4} &= X^{(3)}_{0,1,2,3,4} + \frac{V^{(1)}_{0,3+4,-1-2}\,V^{(1)}_{3+4,3,4}\,V^{(3)}_{-1-2,1,2}}{(\omega_{1+2}+\omega_1+\omega_2)(\omega_{3+4}-\omega_3-\omega_4)} + \frac{V^{(1)}_{0,4-1,3-2}\,V^{(2)}_{4-1,1,4}\,V^{(2)}_{3-2,2,3}}{(\omega_{4-1}+\omega_1-\omega_4)(\omega_{3-2}+\omega_2-\omega_3)} \\ &+ \frac{V^{(1)}_{0,-1-2,3+4}\,V^{(3)}_{-1-2,1,2}\,V^{(1)}_{3+4,3,4}}{(\omega_{1+2}+\omega_1+\omega_2)(\omega_{3+4}-\omega_3-\omega_4)} + \frac{V^{(2)}_{0,1+2,3+4}\,V^{(1)}_{1+2,1,2}\,V^{(1)}_{3+4,3,4}}{(\omega_{1+2}-\omega_1-\omega_2)(\omega_{3+4}-\omega_3-\omega_4)}\end{aligned}$$

$$+\frac{V^{(2)}_{0,\,1-3,\,4-2}\,V^{(2)}_{1-3,\,3,\,1}\,V^{(2)}_{4-2,\,2,\,4}}{(\omega_{1-3}+\omega_3-\omega_1)\,(\omega_{4-2}+\omega_2-\omega_4)}+\frac{V^{(2)}_{0,\,-3-4,\,-1-2}\,V^{(3)}_{-3-4,\,3,\,4}\,V^{(3)}_{-1-2,\,1,\,2}}{(\omega_{3+4}+\omega_3+\omega_4)\,(\omega_{1+2}+\omega_1+\omega_2)}$$

$$+\frac{V^{(3)}_{0,\,1+2,\,-3-4}\,V^{(1)}_{1+2,\,1,\,2}\,V^{(3)}_{-3-4,\,3,\,4}}{(\omega_{1+2}-\omega_1-\omega_2)\,(\omega_{3+4}+\omega_3+\omega_4)}+\frac{V^{(3)}_{0,\,1-3,\,2-4}\,V^{(2)}_{1-3,\,3,\,1}\,V^{(2)}_{2-4,\,4,\,2}}{(\omega_{1-3}+\omega_3-\omega_1)\,(\omega_{2-4}+\omega_4-\omega_2)}$$

$$+\frac{V^{(3)}_{0,\,-3-4,\,1+2}\,V^{(3)}_{-3-4,\,3,\,4}\,V^{(1)}_{1+2,\,1,\,2}}{(\omega_{1+2}-\omega_1-\omega_2)\,(\omega_{3+4}+\omega_3+\omega_4)}-\frac{V^{(1)}_{0,\,3,\,4-1-2}\,\tilde{T}^{(3)}_{4-1-2,\,1,\,2,\,4}}{\omega_{4-1-2}+\omega_1+\omega_2-\omega_4}$$

$$-\frac{V^{(1)}_{0,\,4-1-2,\,3}\,\tilde{T}^{(3)}_{4-1-2,\,1,\,2,\,4}}{\omega_{4-1-2}+\omega_1+\omega_2-\omega_4}-\frac{V^{(2)}_{0,\,1,\,4+3-2}\,\tilde{T}^{(2)}_{4+3-2,\,2,\,4,\,3}}{\omega_{4+3+2}+\omega_2-\omega_3-\omega_4}-\frac{V^{(2)}_{0,\,1+2-3,\,4}\,\tilde{T}^{(2)}_{1+2-3,\,3,\,1,\,2}}{\omega_{1+2-3}+\omega_3-\omega_1-\omega_2}$$

$$-\frac{V^{(3)}_{0,\,1,\,2-3-4}\,\tilde{T}^{(3)}_{2-3-4,\,3,\,4,\,2}}{\omega_{2-3-4}+\omega_3+\omega_4-\omega_2}-\frac{V^{(3)}_{0,\,1-3-4,\,2}\,\tilde{T}^{(3)}_{1-3-4,\,3,\,4,\,1}}{\omega_{1-3-4}+\omega_3+\omega_4-\omega_1}-$$

$$-\frac{V^{(3)}_{-1-2,\,1,\,2}\,W^{(1)}_{0,\,-1-2,\,4,\,3}}{\omega_{1+2}+\omega_1+\omega_2}$$

$$-\frac{V^{(3)}_{-1-2,\,1,\,2}\,W^{(1)}_{0,\,4,\,-1-2,\,3}}{\omega_{1+2}+\omega_1+\omega_2}-\frac{V^{(3)}_{-1-2,\,1,\,2}\,W^{(1)}_{0,\,3,\,4,\,-1-2}}{\omega_{1+2}+\omega_1+\omega_2}-\frac{V^{(1)}_{1+2,\,1,\,2}\,W^{(2)}_{0,\,1+2,\,4,\,3}}{\omega_{1+2}-\omega_1-\omega_2}$$

$$-\frac{V^{(2)}_{4-2,\,2,\,4}\,W^{(2)}_{0,\,1,\,4-2,\,3}}{\omega_{4-2}+\omega_2-\omega_4}-\frac{V^{(2)}_{4-2,\,2,\,4}\,W^{(2)}_{0,\,1,\,3,\,4-2}}{\omega_{4-2}+\omega_2-\omega_4}-\frac{V^{(2)}_{1-4,\,4,\,1}\,W^{(3)}_{0,\,1-4,\,2,\,3}}{\omega_{1-4}+\omega_4-\omega_1}$$

$$-\frac{V^{(2)}_{2-4,\,4,\,2}\,W^{(3)}_{0,\,1,\,2-4,\,3}}{\omega_{2-4}+\omega_4-\omega_2}-\frac{V^{(1)}_{3+4,\,4,\,3}\,W^{(3)}_{0,\,1,\,2,\,3+4}}{\omega_{3+4}-\omega_3-\omega_4}-\frac{V^{(3)}_{-4-3,\,4,\,3}\,W^{(4)}_{0,\,-4-3,\,2,\,1}}{\omega_{4+3}+\omega_4+\omega_3}$$

$$-\frac{V^{(3)}_{-4-3,\,4,\,3}\,W^{(4)}_{0,\,1,\,-4-3,\,2}}{\omega_{4+3}+\omega_3+\omega_4}-\frac{V^{(3)}_{-3-4,\,4,\,3}\,W^{(4)}_{0,\,1,\,2,\,-3-4}}{\omega_{3+4}+\omega_3+\omega_4}. \qquad (\text{A }6b)$$

REFERENCES

COKELET, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. R. Soc. Lond.* A **286**, 183–230.

CRAWFORD, D. R., LAKE, B. M., SAFFMAN, P. G. & YUEN, H. C. 1981 Stability of weakly nonlinear deep-water waves in two and three dimensions. *J. Fluid Mech.* **105**, 177–191.

DAVEY, A. & STEWARTSON, K. 1974 On three-dimensional packets of surface waves. *Proc. R. Soc. Lond.* A **338**, 101–110.

DYSTHE, K. B. 1980 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.

HASSELMANN, K. 1962 On the nonlinear energy transfer in a gravity wave spectrum. Part 1. General theory. *J. Fluid Mech.* **12**, 481–500.

HERTERICH, K. & HASSELMANN, K. 1980 A similarity relation for the nonlinear energy transfer in a finite-depth gravity-wave spectrum. *J. Fluid Mech.* **97**, 215–224.

IUSIM, R. & STIASSNIE, M. 1982 Note on a modification of the nonlinear Schrödinger equation for waves moving over an uneven bottom. *Progress Rep. Dept Civ. Engng Technion.*

JANSSEN, P. A. E. M. 1983 On a fourth-order envelope equation for deep-water waves. *J. Fluid Mech.* **126**, 1–11.

LONGUET-HIGGINS, M. S. 1976 On the nonlinear transfer of energy in the peak of a gravity-wave spectrum: a simplified model. *Proc. R. Soc. Lond.* A **347**, 311–328.

LONGUET-HIGGINS, M. S. 1978 The instabilities of gravity waves of finite amplitude in deep water. II. Subharmonics. *Proc. R. Soc. Lond.* A **360**, 489–505.

LONGUET-HIGGINS, M. S. & PHILLIPS, O. M. 1962 Phase velocity effects in tertiary wave interactions. *J. Fluid Mech.* **12**, 333–336.

McLean, J. W. 1982*a* Instabilities of finite-amplitude water waves. *J. Fluid Mech.* **114**, 315–330.

McLean, J. W. 1982*b* Instabilities of finite-amplitude gravity waves on water of finite depth. *J. Fluid Mech.* **114**, 331–341.

Skjelbreia, L. & Hendrickson, J. 1961 Fifth order gravity wave theory. In *Proc. 7th Conf. Coastal Engng*, vol. 1, pp. 184–196.

Stiassnie, M. 1984 Note on the modified nonlinear Schrödinger equation for deep water waves. *Wave Motion* (in press).

Su, M.-Y. 1982 Three-dimensional deep-water waves. Part 1. Experimental measurement of skew and symmetric wave patterns. *J. Fluid Mech.* **124**, 73–108.

Su, M.-Y., Bergin, M., Marler, P. & Myrick, R. 1982 Experiments on nonlinear instabilities and evolution of steep gravity-wave trains. *J. Fluid Mech.* **124**, 45–72.

Yuen, H. C. & Lake, B. M. 1982 Nonlinear dynamics of deep-water gravity waves. *Adv. Appl. Mech.* **22**, 67–229.

Zakharov, V. E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* (Engl. Transl.) **9**, 190–194.

Zakharov, V. E. & Kharitonov, V. G. 1970 Instability of monochromatic waves on the surface of a liquid of arbitrary depth. *J. Appl. Mech. Tech. Phys.* (Engl. Transl.) **11**, 741–751.