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NOTE ON THE MODIFIED NONLINEAR SCHRÖDINGER EQUATION FOR DEEP WATER WAVES

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It is shown that Zakharov's integral equation yields the modified Schrödinger equation for the particular case of a narrow spectrum.

Recent studies of nonlinear dynamics of deep water gravity waves indicate that the cubic Schrödinger equation seems inadequate even for describing the evolution of weakly nonlinear waves, see Yuen and Lake [5].

Two more accurate descriptions have been proposed, so-far. Dysthe [3] took the perturbation analysis originally used for the derivation of the cubic Schrödinger equation one step further, to *fourth order* in the wave steepness, and derived the so-called Modified Schrödinger equation. Crawford *et al.* [1] proposed an integral equation, first derived by Zakharov [6]. This integral equation, which they called the Zakharov equation, was obtained by an expansion to *third order* in the same wave steepness but in the Fourier space.

The present author believes that the scope of applications of Zakharov's eq. is much wider than that of the Modified Schrödinger eq. and is still far from being exhausted. The limited goal of this note is to show that the Modified Schrödinger eq. is merely a particular-case of the much more general Zakharov eq. In order to achieve this goal, the derivation of the Modified Schrödinger eq. from Zakharov's eq. is outlined in the sequel. This derivation follows the lines of Zakharov [6], used in his derivation of the cubic Schrödinger equation. Zakharov's eq.

$$\mathrm{i}\frac{\partial B}{\partial t}(\boldsymbol{k}, t) = \iiint_{-\infty}^{+\infty} T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) B^*(\boldsymbol{k}_1, t) B(\boldsymbol{k}_2, t) B(\boldsymbol{k}_3, t)$$

$$\times \delta(\boldsymbol{k}+\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3) \exp\{\mathrm{i}[\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)-\omega(\boldsymbol{k}_3)]t\}\, \mathrm{d}\boldsymbol{k}_1\, \mathrm{d}\boldsymbol{k}_2\, \mathrm{d}\boldsymbol{k}_3, \tag{1}$$

is related to the free surface $\eta(\boldsymbol{x}, t)$ by the expression

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|\boldsymbol{k}|}{2\omega(\boldsymbol{k})}\right)^{1/2}\{B(\boldsymbol{k}, t)\exp\{\mathrm{i}[\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t]\}+*\}\, \mathrm{d}\boldsymbol{k}. \tag{2}$$

$*$ denotes the complex conjugate, $\boldsymbol{k}=(k, l)$ is the wave vector, $\boldsymbol{x}=(x, y)$ is the horizontal spatial vector, and ω is the linearized wave frequency related to $\boldsymbol{k}$ through the linear dispersion relation $\omega(\boldsymbol{k})=(g|\boldsymbol{k}|)^{1/2}$

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with g the acceleration of gravity. $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is a rather lengthy real scalar function, given in the appendix of Crawford *et al.* [2]. In order to obtain the Modified Schrödinger eq., the discussion is restricted to narrow spectra, with energy concentrated around $\boldsymbol{k} = \boldsymbol{k}_0 = (k_0, 0)$, and all wave-numbers are rewritten as

$$\boldsymbol{k} = \boldsymbol{k}_0 + \boldsymbol{\chi}, \qquad \boldsymbol{\chi} = (\chi, \lambda), \qquad |\boldsymbol{\chi}|/k_0 = o(1).$$

A new variable $A(\boldsymbol{\chi}, t) = B(\boldsymbol{k}, t)\exp\{-i[\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_0)]t\}$ is introduced in eqs. (1) and (2)

$$i\frac{\partial A}{\partial t}(\boldsymbol{\chi}, t) - [\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_0)]A(\boldsymbol{\chi}, t)$$
$$= \iiint_{-\infty}^{\infty} T(\boldsymbol{k}_0+\boldsymbol{\chi}, \boldsymbol{k}_0+\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_2, \boldsymbol{k}_0+\boldsymbol{\chi}_3)$$
$$\cdot A^*(\boldsymbol{\chi}_1)A(\boldsymbol{\chi}_2)A(\boldsymbol{\chi}_3)\delta(\boldsymbol{\chi}+\boldsymbol{\chi}_1-\boldsymbol{\chi}_2-\boldsymbol{\chi}_3)\,d\boldsymbol{\chi}_1\,d\boldsymbol{\chi}_2\,d\boldsymbol{\chi}_3, \tag{3}$$

$$\eta(\boldsymbol{x}, t) = \frac{1}{2\pi}\left(\frac{\omega(\boldsymbol{k}_0)}{2g}\right)^{1/2}\left\{e^{i[k_0x-\omega(\boldsymbol{k}_0)t]}\int_{-\infty}^{\infty}\left(1+\frac{\chi}{4k_0}\right)A(\boldsymbol{\chi}, t)\,e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}\,d\boldsymbol{\chi} + *\right\}. \tag{4}$$

The last equation is rewritten as follows

$$\eta(\boldsymbol{x}, t) = \mathrm{Re}\{a(\boldsymbol{x}, t)\,e^{i[k_0x-\omega(\boldsymbol{k}_0)t]}\}, \tag{5}$$

where $a(\boldsymbol{x}, t)$, is given by the following Fourier transform

$$a(\boldsymbol{x}, t) = \left(\frac{2\omega(\boldsymbol{k}_0)}{g}\right)^{1/2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1+\frac{\chi}{4k_0}\right)A(\boldsymbol{\chi}, t)\,e^{i\boldsymbol{\chi}\cdot\boldsymbol{x}}\,d\boldsymbol{\chi}. \tag{6}$$

Note, that the Modified Schrödinger eq. is usually expressed in terms of the complex amplitude $a(\boldsymbol{x}, t)$.

The frequency difference $\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_0)$ on the l.h.s. of eq. (3) is replaced by the Taylor expansion in powers of the spectral width

$$\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}_0) = \frac{1}{2}\left(\frac{g}{k_0}\right)^{1/2}\left[\chi - \frac{\chi^2}{4k_0} + \frac{\lambda^2}{2k_0} + \frac{\chi^3}{8k_0^2} - \frac{3\chi\lambda^2}{4k_0^2} + O\left(\frac{|\boldsymbol{\chi}|^4}{k_0^3}\right)\right] \tag{7}$$

Then, eq. (3) is multiplied by $(2\omega(\boldsymbol{k}_0)/g)^{1/2}(1+\chi/4k_0)$ and its inverse Fourier transform is taken, which yields

$$ia_{,t} + \frac{1}{2}\left(\frac{g}{k_0}\right)^{1/2}\left[ia_{,x} - \frac{1}{4k_0}a_{,xx} + \frac{1}{2k_0}a_{,yy} - \frac{i}{8k_0^2}a_{,xxx} + \frac{3i}{4k_0^2}a_{,xyy}\right]$$
$$= \left(\frac{2\omega(\boldsymbol{k}_0)}{g}\right)^{1/2}\frac{1}{2\pi}\iiint_{-\infty}^{\infty}\left(1+\frac{\chi_2+\chi_3-\chi_1}{4k_0}\right)T(\boldsymbol{k}_0+\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_2, \boldsymbol{k}_0+\boldsymbol{\chi}_3)$$
$$\cdot A^*(\boldsymbol{\chi}_1)A(\boldsymbol{\chi}_2)A(\boldsymbol{\chi}_3)\,e^{i(\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}}\,d\boldsymbol{\chi}_1\,d\boldsymbol{\chi}_2\,d\boldsymbol{\chi}_3. \tag{8}$$

Next, one can show that the Taylor expansion of T, to first order in the spectral width yields

$$T(\boldsymbol{k}_0+\boldsymbol{\chi}_2+\boldsymbol{\chi}_3-\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_1, \boldsymbol{k}_0+\boldsymbol{\chi}_2, \boldsymbol{k}_0+\boldsymbol{\chi}_3)$$
$$= \frac{k_0^3}{4\pi^2}\cdot\left\{1 + \frac{3}{2k_0}(\chi_2+\chi_3) - \frac{(\chi_1-\chi_2)^2}{2k_0|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|} - \frac{(\chi_1-\chi_3)^2}{2k_0|\boldsymbol{\chi}_1-\boldsymbol{\chi}_3|} + O\left(\frac{|\boldsymbol{\chi}|^2}{k_0^2}\right)\right\}. \tag{9}$$

Substituting eq. (9) into the r.h.s. of eq. (8) and integrating it, gives

$$\mathrm{i}a_{,t}+\frac{1}{2}\left(\frac{g}{k_0}\right)^{1/2}\left[\mathrm{i}a_{,x}-\frac{1}{4k_0}a_{,xx}+\frac{1}{2k_0}a_{,yy}-\frac{\mathrm{i}}{8k_0^2}a_{,xxx}+\frac{3\mathrm{i}}{4k_0^2}a_{,xyy}\right]$$
$$=\frac{g}{2\omega(\boldsymbol{k}_0)}\left[k_0^3|a|^2a+\frac{\mathrm{i}k_0^2}{2}a^2a^*_{,x}-3\mathrm{i}k_0^2|a|^2a_{,x}\right]-\frac{k_0^2aI}{4\pi^2}, \tag{10}$$

where

$$I=\iint_{-\infty}^{\infty}\frac{(\chi_1-\chi_2)^2}{|\boldsymbol{\chi}_1-\boldsymbol{\chi}_2|}A^*(\boldsymbol{\chi}_1)A(\boldsymbol{\chi}_2)\,\mathrm{e}^{\mathrm{i}(\boldsymbol{\chi}_2-\boldsymbol{\chi}_1)\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{\chi}_1\,\mathrm{d}\boldsymbol{\chi}_2. \tag{11}$$

Using the convolution theorem and the eighth eq. on page 470 of Jones [4] one can show that

$$I=\left(\frac{g}{2\omega(\boldsymbol{k}_0)}\right)2\pi\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}(|a|^2)\frac{x-\xi}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\mathrm{d}\boldsymbol{\xi}. \tag{12}$$

To complete the exposition we introduce the induced mean flow potential $\Phi(x, y, z, t)$ which has to satisfy Neumann's problem in the lower half space with $\Phi_{,z}(z=0)=\omega(\boldsymbol{k}_0)/2(\partial/\partial x)(|a|^2)$, z being the vertical coordinate. For this potential one can show that

$$\Phi_{,x}(z=0)=\frac{\omega(\boldsymbol{k}_0)}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial\xi}(|a|^2)\frac{\xi-x}{|\boldsymbol{x}-\boldsymbol{\xi}|^3}\mathrm{d}\boldsymbol{\xi},\quad \boldsymbol{\xi}=(\xi_1,\xi_2). \tag{13}$$

Thus from eqs. (12) and (13) one can show that eq. (10) becomes, finally:

$$\mathrm{i}\left(a_{,x}+\frac{2k_0}{\omega_0}a_{,t}\right)-\frac{1}{4k_0}a_{,xx}+\frac{1}{2k_0}a_{,yy}-\frac{\mathrm{i}}{8k_0^2}a_{,xxx}+\frac{3\mathrm{i}}{4k_0^2}a_{,xyy}$$
$$=k_0^3|a|^2a+\frac{\mathrm{i}k_0^2}{2}a^2a^*_{,x}-3\mathrm{i}k_0^2|a|^2a_{,x}+\frac{2k_0^2}{\omega(\boldsymbol{k}_0)}a\Phi_{,x}(z=0), \tag{14}$$

which is the conventional form of the Modified Schrödinger eq. The fact that the fourth order (in the wave steepness) Modified Schrödinger eq. is a particular case of the third order Zakharov eq. is less surprising if one realizes that all the fourth order terms emerge as a result of the narrow spectral width assumption, and none of them is of fourth order in the wave amplitude itself.

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