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Derivation of the nonlinear Schrödinger Equation for shoaling wave-groups

By Michael Stiassnie, Ralph M. Parsons Laboratory, Massachusetts Institute of Technology Cambridge, MA 02139 USA*

1. Introduction

The aim of the present note is to provide a simple mathematical model for shoaling of nonlinear wave-groups, namely: modulated wave trains and wave packets. As an introduction we recall some well-known results obtained by linear wave theory.

The free-surface elevation η for a shoaling linear water-wave field is given by:

$$\eta(x,t) = \int_0^\infty \tilde{a}(h_0,\omega)\cdot \exp i\{\int_{x_\infty}^{x} k(h_0,\omega)\,\mathrm{d}x - \omega t\}\,\mathrm{d}\omega \tag{1.1}$$

where x – is the horizontal coordinate, and t – the time. The wavenumber function k and amplitude spectrum $\tilde{a}$ are given by:

$$k\,th(k\,h_0) = \omega^2/g; \qquad \tilde{a} = \tilde{a}_\infty \cdot [th(kh_0)\cdot(1 + 2kh_0/sh(2kh_0))]^{-1/2},$$

and depend on the water depth $h_0(x)$ and the wave frequency ω. Only real and positive values of k are considered and $\tilde{a}_\infty$ is the infinite-depth complex amplitude spectrum at the reference point x_∞.

The term "wave-groups" means that we have wave-fields with narrow spectra in mind. Thus, first, we choose a frequency ω_0 in the domain $[\omega_1, \omega_N]$ assuming that $(\omega_N - \omega_1)/\omega_0 = o(1)$. Second, we substitute in Eq. (1.1) the expressions:

$$\omega = \omega_0(1+\varepsilon\gamma), \quad \text{where} \quad \varepsilon = o(1) \quad \text{and} \quad \gamma_1 = \frac{\omega_1 - \omega_0}{\varepsilon} = O(1),$$

$$\gamma_N = \frac{\omega_N - \omega_0}{\varepsilon} = O(1).$$

$$k(h_0,\omega) = k_0(h_0) + \frac{\omega_0}{(\partial\omega_0/\partial k_0)}\cdot\gamma\varepsilon + R(h_0,\gamma\varepsilon)\cdot(\gamma\varepsilon)^2,$$

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* On leave from the Department of Civil Engineering. Technion, I.I.T. Haifa 32000, Israel.

where $k_0\, th(k_0 h_0) = \omega_0^2/g$; to obtain:

$$\eta(x,t) = \exp i\{\int_{x_\infty}^{x} k_0\,dx - \omega_0 t\} \cdot \int_{\gamma_1}^{\gamma_N} \tilde{a}(h_0,\gamma)\cdot \exp i\{\omega_0 \gamma \tau + (\gamma\varepsilon)^2 \int_{x_\infty}^{x} R\,dx\}\,d\gamma \tag{1.2}$$

where

$$\tau = \varepsilon\left(\int_{x_\infty}^{x} \frac{dx}{(\partial\omega_0/\partial k_0)} - t\right). \tag{1.3 a}$$

Third, we assume a rather mild depth variation, so that $h_0 = h_0(\xi)$, where

$$\xi = \varepsilon^2 x. \tag{1.3 b}$$

Finally, eq. (1.2) yields

$$\eta = \exp i\{\int_{x_\infty}^{x} k_0\,dx - \omega_0 t\}\cdot \int_{\gamma_1}^{\gamma_N} \tilde{A}(\xi,\gamma)\, e^{i\omega_0\gamma\tau}\,d\gamma \tag{1.4}$$

where $\tilde{A}(\xi,\gamma) = \tilde{a}(h_0,\gamma)\exp i(\gamma^2 \int R(\xi)\,d\xi)$.

The importance of the above results lies in the rather natural way by which the scaled variables τ and ξ, see eq. (1.3), are introduced.

In what follows we will, when necessary, distinguish between two types of boundary conditions at x_∞. The first type, for which the amplitude-spectrum $\tilde{a}_\infty$ is a function of ω in the ordinary sense of the word, corresponds to wave-packets which tend to zero as $|\tau| \to \infty$. The second type has its amplitude-spectrum given by a finite sum of Dirac delta functions ($\tilde{a}_\infty = \sum_{n=1}^{N} a_{n_\infty}\delta(\gamma-\gamma_n)$, where γ_n are rational numbers) and corresponds to modulated wave-trains periodic in τ.

In order to discuss the shoaling problem for nonlinear wave groups, (and thus allow for weakly nonlinear interactions among the various components), we turn in the next section to Whitham's modulation equations.

2. Modulation equations

Considering the 2-D problem of wave-groups propagating over water of slowly varying depth h_0, the following five unknowns are usually chosen as dependent variables: the wave amplitude a, the wave frequency ω, the wave number k, the average water depth h, and the current velocity U. To determine these unknowns we start from Whitham's set of modulation equations Whitham (1974), p. 556. Pseudo-phase consistency relation:

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}[g(h-h_0) + \frac{g\,k}{2\,sh(2\,k\,h_0)}\,a^2 + O(\varepsilon^4)] = 0. \tag{2.1 a}$$

Mass conservation equation:

$$\frac{\partial(h-h_0)}{\partial t}+\frac{\partial}{\partial x}\left[h_0 U+\frac{g k}{2\sigma}a^2+O(\varepsilon^4)\right]=0. \tag{2.1 b}$$

Wave-action conservation equation:

$$\frac{\partial}{\partial t}\left[\frac{a^2}{\sigma}+O(\varepsilon^4)\right]+\frac{\partial}{\partial x}\left[\sigma'\frac{a^2}{\sigma}+O(\varepsilon^4)\right]=0. \tag{2.1 c}$$

Consistency condition:

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x}=0 \tag{2.1 d}$$

where, $\sigma=[g\,k\,t\,h(kh_0)]^{1/2}$ is the linear dispersion relation, $\sigma'=\partial\sigma/\partial k$, and ε is a typical wave steepness, $\varepsilon=O(a\,k)$.

Following Whitham (1974, p. 562) and including higher order dispersive terms which arise from the quadratic part of the Lagrangian $G=\omega-\sigma$ (see Whitham p. 526), the dispersion relation is given by:

$$\omega=\sigma+k\,U+\frac{g\,k^2}{2\,\sigma\,c\,h^2(k\,h_0)}(h-h_0)+\frac{g\,k^3\,D}{2\,\sigma}a^2-\frac{\sigma''}{2\,a}\frac{\partial^2 a}{\partial x^2}+O(\varepsilon^4) \tag{2.1 e}$$

where $D=(9\,t\,h^4(k\,h_0)-10\,t\,h^2(k\,h_0)+9)/8\,t\,h^3(k\,h_0)$.

Here, and in what follows, we assume that the small modulation parameter, which is introduced via the boundary condition at x_∞ (see ε in the previous section), is of the same order of magnitude as the typical wave steepness. We allow for arbitrary total depth changes, but require mild bottom slopes, of the order of ε^2 at most.

3. Induced mean flow

To make the variation explicit and to facilitate the derivation we introduce the same multiple scale variables, τ and ξ, as used by Djordjevic' and Redekopp (1978), see also eqs. (1.3),

$$\tau=\varepsilon\left[\int_{x_\infty}^{x}\frac{\mathrm{d}x}{\Omega'(\xi)}-t\right],\qquad \xi=\varepsilon^2 x. \tag{3.1 a, b}$$

Where Ω' is an averaged group velocity, to be defined in the sequel.

Rewriting eqs. (2.1 c) and (2.1 d) with the new coordinates (eqs. 3.1), and averaging them over τ gives:

$$\overline{\frac{\sigma'}{\sigma}a^2}=\text{const}_1=B;\quad \bar{\omega}=\text{const}_2=\Omega. \tag{3.2 a, b}$$

Here we assume that the behavior of the solution as a function of τ is the same as that of the linear boundary condition at x_∞. Namely, decaying for $|\tau| \to \infty$ in the case of wave-packets and with a constant period in the case of modulated wave-trains.

The bars indicate averaging over the appropriate domain in τ (finite for modulated wave-trains and infinite for wave packets) and B, Ω are the averaged wave-action flux and the so-called carrier frequency, respectively. Note that for wave-packets $B = 0$. We also define K, the carrier wave-number:

$$K\,th(K h_0) = \Omega^2/g. \tag{3.3}$$

Rewriting eqs. (2.1 a) and (2.1 b) with the new independent variables, yields:

$$\frac{\partial}{\partial\tau}\left\{\frac{1}{\Omega'}\left[g(h-h_0)+\frac{gk}{2sh(2kh_0)}a^2\right]-U\right\} + \varepsilon\frac{\partial}{\partial\xi}\left\{g(h-h_0)+\frac{gka^2}{2sh(2kh_0)}\right\}=0, \tag{3.4 a}$$

$$\frac{\partial}{\partial\tau}\left\{\frac{1}{\Omega'}\left[h_0U+\frac{gk}{2\sigma}a^2\right]-(h-h_0)\right\}+\varepsilon\frac{\partial}{\partial\xi}\left\{h_0U+\frac{gk}{2\sigma}a^2\right\}=0. \tag{3.4 b}$$

Neglecting, for the time being, the second terms in the above equations, we obtain:

$$h-h_0=-\left(\frac{gkh_0}{2sh(2kh_0)}+\frac{gk\Omega'}{2\sigma}\right)\frac{a^2}{gh_0-(\Omega')^2}+\Omega'\frac{h_0G_1+\Omega'G_2}{gh_0-(\Omega')^2}, \tag{3.5 a}$$

$$U=-\left(\frac{g^2k}{2\sigma}+\frac{gk\Omega'}{2sh(2kh_0)}\right)\frac{a^2}{gh_0-(\Omega')^2}+\Omega'\frac{gG_2+\Omega'G_1}{gh_0-(\Omega')^2} \tag{3.5 b}$$

where, G_1 and G_2 are functions of ξ, which emerged as a result of the integration. Now, averaging eqs. (3.4) and substituting (3.5) yields:

$$\left[\frac{gK}{2sh(2Kh_0)}-\frac{g}{gh_0-(\Omega')^2}\left(\frac{gKh_0}{2sh(2Kh_0)}+\frac{gK\Omega'}{2\Omega}\right)\right]\overline{a^2} + g\Omega'\frac{h_0G_1+\Omega'G_2}{gh_0-(\Omega')^2}=\mathrm{const}_3, \tag{3.6 a}$$

$$\left[\frac{gK}{2\Omega}-\frac{h_0}{gh_0-(\Omega')^2}\left(\frac{g^2K}{2\Omega}+\frac{gK\Omega'}{2sh(2Kh_0)}\right)\right]\overline{a^2} + h_0\Omega'\frac{gG_2+\Omega'G_1}{gh_0-(\Omega')^2}=\mathrm{const}_4 \tag{3.6 b}$$

where, consistent with the order of approximation k and σ have been replaced by K, Ω.

Following Stiassnie and Peregrine (1980) we assume zero averaged mass flow (thus, $\text{const}_4 = 0$) and choose such a reference level that $\overline{h - h_0} = 0$ in deep water (which, in turn, sets $\text{const}_3 = 0$). Having fixed these tow constants we solve eqs. (3.6) for G_1 and G_2 and then return to eqs. (3.5) to obtain the final results for the induced mean flow:

$$U(\tau, \xi) = \left(\frac{g^2 K}{2\Omega} + \frac{g K \Omega'}{2 sh(2 K h_0)}\right) \cdot \frac{\overline{a^2} - a^2}{g h_0 - (\Omega')^2} - \frac{g K \overline{a^2}}{2 \Omega h_0}, \tag{3.7}$$

$$h(\tau, \xi) - h_0(\xi) = \left(\frac{g K h_0}{2 sh(2 K h_0)} + \frac{g K \Omega'}{2\Omega}\right) \cdot \frac{\overline{a^2} - a^2}{g h_0 - (\Omega')^2} - \frac{K \overline{a^2}}{2 sh(2 K h_0)}. \tag{3.8}$$

Consistent with our level of approximation we have

$$\overline{a^2} = \frac{\Omega}{\Omega'} B = \begin{cases} \dfrac{g}{2\Omega\Omega'} \overline{a_\infty^2}, & \text{for modulated wave-trains,} \\ 0, & \text{for wave-packets.} \end{cases} \tag{3.9}$$

A simple example, showing the calculations of $\overline{a_\infty^2}$ and Ω, is given in appendix A.

4. Nonlinear Schrödinger equation

To derive the NLS we rewrite eq. (2.1 c) as follows

$$\frac{2a}{\sigma}\frac{\partial a}{\partial t} - \frac{\sigma' a^2}{\sigma}\frac{\partial k}{\partial t} + \frac{2\sigma' a}{\sigma}\frac{\partial a}{\partial x} - \frac{\sigma' a^2}{\sigma^2}\frac{\partial \sigma}{\partial x} + \frac{a^2}{\sigma}\frac{\partial \sigma'}{\partial x} = 0. \tag{4.1}$$

Applying eq. (2.1 d), which gives mutual cancellation of the second and fourth terms in the above equation, and dividing by $2a/\sigma$ yields

$$\frac{\partial a}{\partial t} + \sigma' \frac{\partial a}{\partial x} + \frac{a}{2}\frac{\partial \sigma'}{\partial x} = 0. \tag{4.2}$$

The Taylor series expansion of $\sigma(k)$ in the vicinity of $k = K$ is

$$\sigma = \Omega + \Omega' \cdot (k - K) + \frac{\Omega''}{2} \cdot (k - K)^2 + o(\varepsilon^2). \tag{4.3}$$

Using this series we rewrite eq. (4.2) as well as the dispersion relation (2.1 e):

$$\frac{\partial a}{\partial t} + \Omega' \frac{\partial a}{\partial x} + \Omega'' \cdot (k - K)\frac{\partial a}{\partial x} + \frac{\Omega'' a}{2}\frac{\partial (k - K)}{\partial x} + \frac{a}{2}\frac{\partial \Omega'}{\partial x} = 0, \tag{4.4a}$$

$$\omega = \Omega + \Omega' \cdot (k - K) + \frac{\Omega''}{2} \cdot (k - K)^2 + \alpha_1 a^2 - \frac{\Omega''}{2a}\frac{\partial^2 a}{\partial x^2} + \alpha_2 \overline{a^2}. \tag{4.4b}$$

Here

$$\alpha_1 = \frac{g K^3}{2\Omega} \cdot \frac{9 th^4(K h_0) - 10 th^2(K h_0) + 9}{8 th^3(K h_0)} - \left(\frac{g^2 K}{2\Omega} + \frac{g K \Omega'}{2 sh(2 K h_0)}\right) \cdot \frac{K}{g h_0 - (\Omega')^2} - \left(\frac{g K h_0}{2 sh(2 K h_0)} + \frac{g k \Omega'}{2\Omega}\right) \cdot \frac{g K^2}{2\Omega [g h_0 - (\Omega')^2] c h^2(K h_0)}, \tag{4.5a}$$

$$\alpha_2 = -\alpha_1 - \frac{g k^2}{2\Omega h_0} - \frac{g K^3}{4\Omega sh(2 K h_0) ch^2(K h_0)} + \frac{g K^3}{2\Omega} \cdot \frac{9 th^4(K h_0) - 10 th^2(K h_0) + 9}{8 th^3(K h_0)}. \tag{4.5b}$$

Referring to eq. (2.1 d) we define a phase function θ so that

$$\omega - \Omega = -\frac{\partial\theta}{\partial t}, \quad k - K = \frac{\partial\theta}{\partial x}. \tag{4.6}$$

Substituting eq. (4.6) into eqs. (4.4 a) and (4.4 b) we obtain

$$\frac{\partial a}{\partial t} + \Omega' \frac{\partial a}{\partial x} + \frac{\Omega''}{2}\left(a \frac{\partial^2\theta}{\partial x^2} + 2 \frac{\partial\theta}{\partial x}\frac{\partial a}{\partial x}\right) + \frac{a}{2}\frac{\partial\Omega'}{\partial x} = 0, \tag{4.7a}$$

$$\frac{\partial\theta}{\partial t} + \Omega' \frac{\partial\theta}{\partial x} + \frac{\Omega''}{2}\left(\left(\frac{\partial\theta}{\partial x}\right)^2 - \frac{1}{a}\frac{\partial^2 a}{\partial x^2}\right) + \alpha_1 a^2 + \alpha_2 \overline{a^2} = 0. \tag{4.7b}$$

Multiplying eq. (4.7 a) by the imaginary unit i, adding to it $(-a)$ times eq. (4.7 b) and then multiplying the sum by $e^{i\theta}$ we get

$$\frac{i}{2}\left(\frac{\partial\Omega'}{\partial x}\right) A + i(A_{,t} + \Omega' A_{,x}) + \frac{\Omega''}{2} A_{,xx} - \alpha_1 |A|^2 A = \alpha_2 \overline{a^2} A \tag{4.8}$$

where $A = a e^{i\theta}$ is a complex wave envelope. Alternatively, using the scaled coordinates (3.1 a, b) we have

$$\frac{i}{2\Omega'}\frac{\partial\Omega'}{\partial\xi} A + i A_{,\xi} + \frac{\Omega''}{2(\Omega')^3} A_{,\tau\tau} - \frac{\varepsilon^{-2}\alpha_1}{\Omega'} |A|^2 A = \frac{\alpha^2}{\Omega'} \varepsilon^{-2} \overline{a^2} A. \tag{4.9}$$

The last equation is almost the same as that obtained by Djordjevic' and Redekopp (1978) (Note that their A is $g/2\Omega\varepsilon$ times the one in eq. 4.9). In principle, after solving eq. (4.9) one can go back to eqs. (3.7) and (3.8) and obtain the induced mean current velocity $U(\tau, \xi)$ and the mean free surface $h(\tau, \xi) - h_0(\xi)$ right away. These induced mean flow quantities are of great practical interest, since they are probably related to such phenomena as surf beats, long shore cellular structure, and harbor resonance.

5. Comparison with Djordjevic' and Redekopp

Djordjevic' and Redekopp used a multiple scale method and started their derivation from the Laplace equation for the velocity potential and the non-linear free-surface boundary condition. Their approach is basicly the same as the one used by Davey and Stewartson (1974) for water of constant depth. Equation (4.9) has a known function of ξ on its r.h.s. whereas previous authors did not attempt to determine this term explicitly (except for wave-packets, for which it is identically zero). The way in which eq. (4.9) enables to obtain the well-known monochromatic wave-train solution is shown in appendix B.

In order to obtain eq. (4.9) from Djordjevic' and Redekopp's results we start from their eqs. (2.11) and (2.12) given, in our notation, by:

$$\frac{i}{2\Omega'}\frac{\partial\Omega'}{\partial\xi}A + iA_{,\xi} + \frac{\Omega''}{2(\Omega')^3}A_{,\tau\tau} - \frac{\varepsilon^{-2}\beta_1}{\Omega'}|A|^2 A$$

$$= \frac{\beta_2}{\Omega'}\varphi_{10,\tau}A + \frac{\beta_3}{\Omega'}\tilde{\varphi}(\xi)A, \tag{5.1}$$

$$\varphi_{10,\tau} = \frac{\varepsilon^{-2}g^2\beta_2|A|^2}{2\Omega[1 - g h_0/(\Omega')^2]} + Q(\xi) \tag{5.2}$$

where

$$\beta_1 = \frac{gK^3}{2\Omega}\,\frac{9 - 12\,th^2(Kh_0) + 13\,th^4(Kh_0) - 2\,th^6(Kh_0)}{8\,th^3(Kh_0)}, \tag{5.3.a}$$

$$\beta_2 = \frac{K^2}{2\Omega}\left(\frac{2\Omega}{K\Omega'} + \operatorname{sech}^2(Kh_0)\right), \tag{5.3.b}$$

$$\beta_3 = \frac{K^2}{2\Omega}\operatorname{sech}^2(Kh_0). \tag{5.3.c}$$

Note that the last term on the r.h.s. of eq. (5.1) does not appear in Djordjevic' and Redekopp. In the sequel, we will show the necessity of including the term $\varepsilon^2\tilde{\varphi}(\xi)\cdot t$ in the wave-induced mean current velocity potential., φ_0, which is, thereupon, given by:

$$\varphi_0 = \varepsilon\varphi_{10}(\tau,\xi) + \varepsilon^2\tilde{\varphi}(\xi)\cdot t + \varepsilon^2\varphi_{20}(\tau,\xi) + O(\varepsilon^3). \tag{5.4}$$

To second order in ε, the wave induced mean current velocity is given by

$$U = \varepsilon\varphi_{10,x} = \frac{g^2\beta_2|A|^2}{2\Omega\Omega'[1 - g h_0/(\Omega')^2]} + \frac{\varepsilon^2 Q}{\Omega'}. \tag{5.5}$$

To find Q, we impose a lateral boundary condition of zero averaged (over τ) mass flow, which is appropriate for an impervious beach, as follows:

$$h_0\bar{U} + \frac{gK}{2\Omega}\overline{a^2} = 0. \tag{5.6}$$

From eqs. (5.6) and (5.5) we obtain

$$Q(\xi) = -\varepsilon^{-2}\overline{a^2}\left\{\frac{gK\Omega'}{2\Omega h_0} + \frac{g^2\beta_2}{2\Omega[1 - g h_0/(\Omega')^2]}\right\}. \tag{5.7}$$

Substituting eq. (5.7) into eq. (5.5) we recover eq. (3.7).

Integration of eq. (5.2) with respect to τ yields

$$\varphi_{10} = \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega[1 - g h_0/(\Omega')^2]}\int |A|^2\,d\tau + Q(\xi)\tau + \varepsilon P(\xi). \tag{5.8}$$

The first and second terms in the above equation grow monotonically, and boundlessly with time. Secular terms of this nature are bound to cause troubles in higher order derivation and should be suppressed. The addition of the term $\varepsilon^2\tilde{\varphi}(\xi)t$ to eq. (5.4) seems to be the proper way to achieve this goal. Substituting eq. (5.8) into (5.4) gives:

$$\varphi_0 = \frac{g^2\beta_2}{2\Omega[1 - g h_0/(\Omega')^2]}\left\{\int_{x_0}^{x}\frac{|A|^2\,dx}{\Omega'} - \int_0^t (|A|^2 - \overline{a^2})\,dt - \overline{a^2}\,t\right\}$$
$$+ \varepsilon^2 Q\left\{\int_{x_0}^{x}\frac{dx}{\Omega'} - t\right\} + \varepsilon^2 P + \varepsilon^2\tilde{\varphi}t + \varepsilon^2\varphi_{20}. \tag{5.9}$$

Thus, suppressing secular terms in t we get:

$$\tilde{\varphi} = \frac{\varepsilon^{-2}g^2\beta_2\overline{a^2}}{2\Omega[1 - g h_0/(\Omega')]} + Q = -\frac{\varepsilon^{-2}gK\Omega'\overline{a^2}}{2\Omega h_0}. \tag{5.10}$$

Following Djordjevic' and Redekopp's derivation, but including $\tilde{\varphi}$, one can recover eq. (3.8) for the average water surface level. Finally, substitution of eq. (5.7) for Q, and eq. (5.10) for $\tilde{\varphi}$, into eq. (5.1), leads to an equation identical to eq. (4.9). Note that by suppressing secular terms in x one can also show that $P = -\tilde{\varphi}\int_{x_0}^{x} dx/\Omega'$. Anyhow, this seems to be of less importance, since P does not contribute anything to our third order derivation, in contrast to φ which contributes to the lowest order average water surface level.

6. Simple cases

A simpler and dimensionless form of eq. (4.9) is obtained by means of the transformations

$$\psi = \varepsilon^{-1}\left(\frac{2\Omega^5\Omega'}{g^3}\right)^{1/2} A\cdot\exp i\left(\overline{a^2}\int_{x_\infty}^{x}\frac{\alpha_2}{\Omega'}\,dx\right), \tag{6.1 a}$$

$$T = \Omega\tau = \varepsilon\Omega\left(\int_{x_\infty}^{x}\frac{dx}{\Omega'} - t\right), \tag{6.1 b}$$

$$X = \Omega^2 \int_{\xi_\infty}^{\xi} \frac{\Omega''}{2(\Omega')^3} \, d\xi = \varepsilon^2 \Omega^2 \int_{x_\infty}^{x} \frac{\Omega''}{2(\Omega')^3} \, dx, \tag{6.1c}$$

which yield

$$i\psi,_X + \psi,_{TT} + \mu |\psi|^2 \psi = 0; \qquad \mu(X) = \frac{-g^3 \Omega' \alpha_1}{\Omega^7 \Omega''} \tag{6.2}$$

The dimensionless parameter μ is a monotonic increasing function of $K h_0$, having the values zero and one for $K h_0 = 1.363$ and $K h_0 \to \infty$ respectively. The free surface elevation is given by:

$$\eta = \varepsilon \operatorname{Re} \left\{ \psi \exp i \left(\int_{x_\infty}^{x} K \, dx - \Omega t \right) \right\}. \tag{6.3}$$

In order to complete the mathematical statement of the "shoaling wave-group" problem one has to supplement eq. (6.2) with a boundary condition at $X = 0$ ($x = x_\infty$). We suggest to focus the attention to the following two, relatively simple, model problems.

(i) Modulated wave-trains

Here, following the approach of Stiassnie & Kroszynski (1982), we consider a system initially composed of a carrier-wave and a symmetric "side-band" disturbance. The water surface elevation of such a system at $x = x_\infty$ is given by

$$\eta(t, x_\infty) = \varepsilon K_\infty^{-1} \operatorname{Re} \{ e^{-i\Omega t} + \beta e^{-i[(1+\gamma\varepsilon)\Omega t - \alpha]} + \beta e^{-i[(1-\gamma\varepsilon)\Omega t - \alpha]} \}. \tag{6.4}$$

The appropriate boundary condition for ψ is

$$\psi(X = 0) = 1 + 2\beta \varepsilon^{1\alpha} \cos(\gamma T), \tag{6.5}$$

which is periodic in T with period $2\pi\gamma^{-1}$.
Applying the results of Appendix A one finds that $\overline{a_\infty^2} = (\varepsilon K_\infty^{-1})^2 (1 + 2\beta^2)$ for this case. Note also that $\beta \equiv 0$ corresponds to the monochromatic wave-train solution, presented in Appendix B.

(ii) Wave packets

For this case we follow Grimshaw (1979) and take as input a deep water soliton, which is an exact solution of eq. (6.2) for $\mu = 1$. The appropriate boundary condition becomes:

$$\psi(X = 0) = \delta \cdot \operatorname{sech}(\delta T / \sqrt{2}). \tag{6.6}$$

Analytical solutions of these two model problems are currently studied and will be reported in future publications.

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Appendix A: Calculation of $\overline{a_\infty^2}$ and Ω.

Referring to the notation of section 1 we consider the following simple linear-group in infinitely deep water, $N = 3: \gamma_1 = -1$, $\gamma_2 = 0$, $\gamma_3 = 1$, $a_{1\infty} = a_{3\infty} = q$, and $a_{2\infty} = Q$. From eq. (1.2) we have

$$\eta = \exp i\{k_0 x - \omega_0 t\} \cdot \{q\, e^{i(-\omega_0\tau + R)} + Q + q\, e^{i(\omega_0\tau + R)}\} \tag{A.1}$$

periodic in τ with period $2\pi/\omega_0$.
If we rewrite this equation in the form

$$\eta = a_\infty \exp i\{k_0 x - \omega_0 t + \alpha_\infty\} \tag{A.2}$$

we have

$$a_\infty^2 = Q^2 + 4q^2\cos^2(\omega_0\tau) + 4qQ\cos R\, cos(\omega_0\tau); \tag{A.3 a}$$

$$\omega = \omega_0 + \varepsilon\frac{\partial\alpha_\infty}{\partial\tau}; \qquad \alpha_\infty = \mathrm{tg}^{-1}\left\{\frac{2q\sin R\cos(\omega_0\tau)}{Q + 2q\cos R\cos(\omega_0\tau)}\right\} \tag{A.3 b}$$

Averaging over τ yields

$$\overline{a_\infty^2} = Q^2 + 2q^2, \qquad \Omega = \omega_0. \tag{A.4}$$

Appendix B: The monochromatic wave-train solution.

For this case $A = A(\xi)$, $\overline{a^2} \equiv |A|^2$, and from eq. (4.9) we have

$$\frac{1}{2}\cdot\frac{\partial\Omega'}{d\xi}A + i\,\Omega' A_{,\xi} = \varepsilon^{-2}(\alpha_1 + \alpha_2)|A|^2 A. \tag{B.1}$$

Recalling that $A = a\,e^{i\theta}$, the imaginary and real parts of eq. (B.1), respectively, are

$$\frac{1}{2}\cdot\frac{\partial\Omega'}{\partial\xi}a + \Omega'\frac{\partial a}{\partial\xi} = 0, \qquad -\Omega'\frac{\partial\theta}{\partial\xi} = (\alpha_1 + \alpha_2)\varepsilon^{-2}a^2 \tag{B.2 a, b}$$

where $(\alpha_1 + \alpha_2)$ is given in eq. (4.5 b). The solutions of these eqs. are

$$\Omega' a^2 = \text{const}, \qquad k = K - \frac{(\alpha_1 + \alpha_2)}{\Omega'}a^2, \tag{B.3 a, b}$$

as they should be.

References

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Abstract

A nonlinear Schrödinger equation with varying coefficients, describing the evolution of surface wave groups moving over an uneven bottom, is derived from Whitham's modulation equations. The derivation yields new expressions for the wave-induced mean flow field.

Zusammenfassung

Ausgehend von Whitham's Modulationsgleichungen wird eine nichtlineare Schrödinger-Gleichung mit veränderlichen Koeffizienten hergeleitet, welche die Entwicklung von Oberflächen-wellengruppen beschreibt, die sich über einen unebenen Boden fortpflanzen. Die Herleitung ergibt neue Ausdrücke für das wellen-induzierte mittlere Strömungsfeld.

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