Particle Motion under Stokes Waves

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We recall two well documented facts, see Phillips (1). First, the Eulerian mean velocity (averaged over a wave period) at any point below the wave troughs is exactly zero, thus causing zero mean mass flux across any vertical control section between the bottom and a fixed point deeper than the troughs. Second, the Lagrangian mean velocity of all water particles is a nonzero second order quantity, and is always in the wave propagation direction. At first sight, it seems justified to ask, as probably many students do, how can it be that while all individual particles move forward, the mean mass flux across the control section remains zero? In the present note we present, what is hoped to be, a clear and descriptive answer to this question.

In Fig. 1 we show the pathlines of four selected water particles during one wave period, starting from the instant in which the particles near the control section AB are moving upwards. Particle 1 crosses the section AB twice during a wave period and contributes nothing to the mean mass flux. Particle 2, which starts from a location nearer to AB crosses the control section only once, from left to right, thus contributing to a positive mass flux. (The writers define the flux in the wave propagation direction "as positive.") All particles in zone 1, (the zone shaded with horizontal lines) are the same type as particle 2. The width c of this zone is identical with the particle displacement during a wave period and is given, according to Phillips, as follows:

$$c = \pi k \alpha^2 \cdot \frac{ch\,[2k(z+h)]}{sh^2(kh)} \qquad (1)$$

in which α is the wave amplitude, k the wave-number and h the mean water depth.

The area of zone 1, A_1, is given by

$$A_1 = \int_{-h}^{-H} c(z)\,dz = \frac{\pi\alpha^2}{2} \cdot \frac{sh\,[2k(h-H)]}{sh^2(kh)} \qquad (2)$$

where H is the depth of the point B beneath the mean water surface ($H > \alpha$).

It can be shown, accurate to order α^2 for the mass flux, that all particles

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1:1.5 could thus experience damage if the stability tests upon which the design was based did not encompass values of H/L as great as $H/L = 0.05$.

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Appendix.—References

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inside zone 2 (the ellipse shaded with vertical lines), which has semi-axes a and b, have pathlines similar to that of particle 3 and cross AB only once, from right to left. All these particles contribute to a negative mass flux.

The semi-axes a and b are equal to those of the first order elliptic orbits as given by Phillips.

$$a = \alpha \cdot \frac{ch\,[k(h - H)]}{sh\,(kh)}; \quad b = \alpha \cdot \frac{sh\,[k(h - H)]}{sh\,(kh)} \qquad (3)$$

Thus we obtain the area of zone 2 (which is denoted as A_2):

$$A_2 = \pi ab = \frac{\pi\alpha^2}{2} \cdot \frac{sh\,[2k(h - H)]}{sh^2(kh)} \qquad (4)$$

Particles, in the neighborhood of zone 2, that start above $z = -H$ (e.g. particle 4), do not cross AB at all. Particles, in the neighborhood of the ellipse, that start below $z = -H$ behave like particle 1.

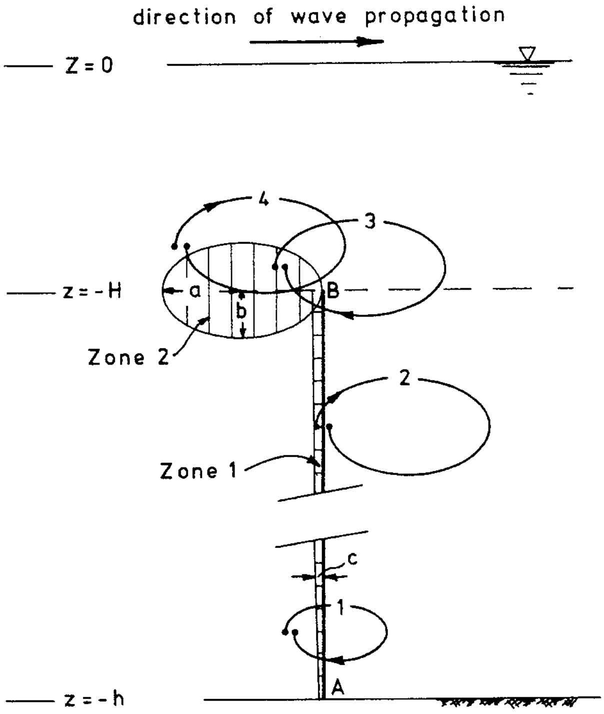

FIG. 1.—Pathlines of Typical Water Particles under Stokes Waves

From Eqs. 2 and 4, we see that the mean mass flux during a wave period, given by $(A_1 - A_2)/T$ in which T is the wave period, equals zero, despite the fact that all particles move to the right.

In order to avoid misunderstandings, note that the choice of any different initial instant, would result in the same discussion and results. The only difference will be the displacement to the right, by a distance not bigger than 2a, of the ellipse (zone 2, in Fig. 1). It is worth mentioning that, when point B is located in the region $z(-\alpha, \alpha)$, the elliptic zone 2, is only partially occupied at the initial instant. Thus the area of the relevant part of zone 2 is decreasing from πab, for B at $z < \alpha$, to zero, for B at $z = \alpha$, causing the mean flux to increase from zero to $\pi\alpha^2/T \coth(kh)$, as should be expected.

A new approach is used in this short note to show that, despite the fact that all individual water particles move forward, the mean mass flux across a vertical control section (from the bottom to a point located below the wave troughs) remains zero.

Appendix.—Reference

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