A simple mathematical model of a floating breakwater

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A simple mathematical model, based on the solution of the two-dimensional problem of a vertical floating plate and on rigid body dynamics, is used to investigate the influence of different characteristics (such as mass, draft and anchoring) on the breakwater performance. The results include information about the transmission coefficient as well as about the plate displacement and anchoring forces, as functions of the plate and incident wave parameters.

INTRODUCTION

Although there exists a great volume of published work dealing with floating breakwaters[4], it appears that there is a lack of a simple mathematical model for these structures. Such a model must determine the influence of various breakwater characteristics (such as mass, draft, mooring stiffness) on its performance (displacements and anchor forces) and upon the transmission coefficient (defined as the transmitted wave to incident wave amplitude ratio). An approach in this direction was made by Adee *et al.*[1] who adopted a numerical, two-dimensional linear model, originally developed for ships, which is applicable to rather general cross-sections.

The floating plate model, proposed in this article, has the advantage of having a closed mathematical solution. It permits focusing on the influence of each parameter separately.

MATHEMATICAL MODEL AND SOLUTION

In order to simplify the problem in a manner that enables analytical treatment, it is proposed to consider a breakwater with a simple form and to adopt some assumptions common in naval hydrodynamics. The chosen breakwater model consists of a vertical thin plate, whose upper edge is above the water surface and the lower one extending to depth T beneath the surface. The breakwater mass per unit breadth is m, and the moment of inertia per unit breadth about the centre of gravity, which is located at depth c beneath the water surface, is I_c.

The plate may float freely or be anchored at depth b to cables, which are represented by linear springs having the constant K per unit breadth. The problem is two-dimensional (in the plane y, z*, see Fig. 1), the water depth is assumed to be infinite and its density ρ constant. Monochromatic waves with frequency σ and amplitude r_0, approach from the left with fronts parallel to the breakwater. Part of the wave energy is reflected by the plate and part is transmitted beneath it. The attacking waves set the breakwater into periodic motion. The horizontal velocity of the point O (intersection of breakwater with water-surface), which is usually called sway, is given by $Re_j[V]$, where $V=v\exp(j\sigma t)$; while the angular velocity about this point, called roll, is given by $Re_j[\Omega]$, $\Omega=\omega\exp(j\sigma t)$, where t is the time.

The breakwater motions generate out-going waves, both up and down stream. Using the assumption of irrotational flow and considering linear waves, there exists a velocity potential $\Phi(y, z, t)$ satisfying the Laplace equation:

$$\Phi_{,yy}+\Phi_{,zz}=0, \qquad z\geqslant 0 \tag{1}$$

and the free-surface boundary condition:

$$\Phi_{,tt}-g\Phi_{,z}=0, \qquad z=0 \tag{2}$$

The free-surface elevation η is given by the expression:

$$\eta=\frac{1}{g}\cdot\Phi_{,t} \qquad z=0 \tag{3}$$

The boundary condition on the plate (after linearization) is:

$$\Phi_{,y}=Re_j[V+\Omega z], \qquad \text{for } y=0 \text{ and } 0\leqslant z\leqslant T \tag{4}$$

where V, Ω are the plate velocities.

The potential of the incoming wave (from $y=-\infty$) is:

$$\Phi_0=Re_j[\varphi_0\exp(j\sigma t)];$$

$$\varphi_0=-(jgr_0/\sigma)\cdot\exp[-\nu(z+jy)] \tag{5}$$

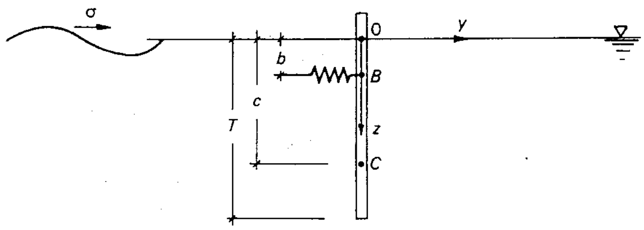

Figure 1. Vertical cross-section of the breakwater

* For the sake of simplicity the notation in the present article follows that of Haskind[3], as far as possible.

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where $\nu = \sigma^2/g = 2\pi/\lambda$ is the wave number and λ the wave length. The solution of the above stated problem is obtained as follows: (i) assuming the velocities V, Ω are known the hydrodynamical problem is solved first. This problem is defined by equation (1) and the boundary conditions (2) and (4) and the radiation-condition (which prohibits energy input, except that connected with the incoming wave (5)). The solution of the hydrodynamical problem enables one to calculate the forces and moments of the fluid on the plate; (ii) the unknown velocities V, Ω are computed from the equations of motion of a rigid body (the breakwater), taking into account the various forces and moments acting on the breakwater; (iii) last, the transmission coefficient and the force in the spring (anchor) are obtained.

The hydrodynamical problem has been solved analytically, using complex-variable techniques, by Haskind[3]. According to Haskind's solution the horizontal hydrodynamical force $Re_j[Y]$ and the hydrodynamical moment, $Re_j[M]$, about the point O are given by the following expressions:

$$Y = [y_g - (j\sigma\mu_{22} + \lambda_{22})v - (j\sigma\mu_{24} + \lambda_{24})\omega]\exp(j\sigma t) \quad (6)$$

$$M = [m_g - (j\sigma\mu_{42} + \lambda_{42})v - (j\sigma\mu_{44} + \lambda_{44})\omega]\exp(j\sigma t) \quad (7)$$

y_g, m_g, which are the results of the scattering of waves by a fixed plate are given by:

$$y_g = -2\rho g r_0 T S_1 \cdot [\pi I_1(\mu) + jK_1(\mu)]/[\pi^2 I_1^2(\mu) + K_1^2(\mu)] \quad (8)$$

$$m_g = -2\rho g r_0 (T^2/\mu) \cdot (S_1 - \pi/4) \cdot [\pi I_1(\mu) + jK_1(\mu)]/[\pi^2 I_1^2(\mu) + K_1^2(\mu)] \quad (9)$$

K_1, I_1 are Bessel functions of the argument $\mu = \nu T$ and the quantity $S_1 = 0.5\pi[I_1(\mu) + L_1(\mu)]/\mu$ depends on the modified Struve function $L_1(\mu)$[2].

The remaining terms in equations (6) and (7) are connected with the radiation of waves from an oscillating plate, in otherwise still water. λ_{nm}, μ_{nm} are generalized damping and added mass coefficients, respectively. The subscript 2 represents the horizontal oscillation and the subscript 4 the angular movement. The coefficients λ_{nm}, μ_{nm} are given in the Appendix.

The equations of motion of the plate are used to determine the velocities V, Ω which remained arbitrary in Haskind's work. The horizontal equation (in the y direction) is:

$$Y + jK(V + b\Omega)/\sigma = m(\dot{V} + c\dot{\Omega}) \quad (10)$$

where the term Y, given by (6), represents the hydrodynamical forces and the second term on the l.h.s. results from the forces applied by the spring. The term on the r.h.s. equals the mass m multiplied by the horizontal acceleration of the centre of gravity, (the dot above the velocities notes differentiation with respect to time).

The equation of the rotational motion about the centre of gravity is:

$$(M - cY) - jK(c-b)(V + b\Omega)/\sigma + jmg(c - T/2)\Omega/\sigma = I_c\dot{\Omega} \quad (11)$$

The first term on the l.h.s. represents the hydrodynamical moment about the centre of gravity and is given by equations (6) and (7), while the second and third terms represent the returning moments of the spring and of the hydrostatic pressures, respectively. The r.h.s. is obtained by multiplying the moment of inertia I_c by the angular acceleration about the point C.

The two equations (10) and (11) form a linear complex algebraic system with two unknowns V and Ω.

Mooring forces and transmission coefficients. The maximum force in the spring is given by the following formula:

$$F = K|v + b\omega|/\sigma \quad (12)$$

According to Haskind the flow potential for $y = \infty$ is:

$$\Phi = Re_j\{(-jgr_0/\sigma + jvB_2 + j\omega B_4 + jB_7)\exp[-\nu(z + jy) + j\sigma t]\} \quad (13)$$

where the subscript 7 denotes the diffraction problem. Taking into account equations (3) and (5) the transmission coefficient is found to be:

$$T_c = |1 - \sigma(vB_2 + \omega B_4 + B_7)/(gr_0)| \quad (14)$$

where B_2, B_4 and B_7 are given in the Appendix.

INPUT AND OUTPUT DATA

Input data

Despite the fact that the configuration of the floating breakwater suggested herein is almost the simplest possible, it is required, at least at first sight, to specify ten different input quantities. These are given in the following table:

wave data:
- λ wave length of the incident wave
- r_0 amplitude of the incident wave
- ρ water density
- g acceleration of gravity

breakwater data:
- T draft
- c depth of the centre of gravity
- m mass per unit breadth
- I_c moment of inertia per unit breadth

mooring data:
- K spring constant per unit breadth
- b depth of mooring point

In order to reduce the number of parameters we limit the discussion to homogeneous plates the density of which is one half of the water density and with a thickness δ.

The mass per unit breadth of such a plate is $\rho\delta T/2$. An

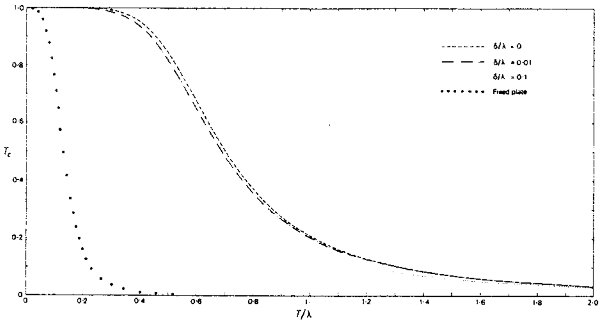

Figure 2. The influence of the mass parameter on the transmission coefficient (K = 0)

additional concentrated mass per unit breadth, $\rho\delta T/2$ in magnitude, is set at the lower end of the plate. Thus, the total mass is equal to the mass of the water displaced by the body:

$$m=\rho\delta T \tag{15}$$

The centre of gravity of the breakwater is therefore at the depth:

$$c=0.75\ T \tag{16}$$

which guarantees stability of floatation.

The moment of inertia per unit breadth of this plate is:

$$I_c=5\rho\delta T^3/48 \tag{17}$$

The above assumptions reduce the number of breakwater data parameters from four (T, c, m, I_c) to two (T, δ) putting the overall number of input data parameters on (8).

Next, we choose λ, $(\lambda/g)^{1/2}$ and $\rho\lambda^3$ as length, time and mass scales respectively and switch to non-dimensional variables. We also recall that the wave steepness parameter, r_0/λ, is only weakly involved in the computation because of the linearity assumption.

As a result, the original set of ten quantities is reduced to a set of four most relevant non-dimensional input parameters, namely, a geometrical parameter T/λ, a mass parameter δ/λ, a mooring stiffness parameter $K/\rho g\lambda$ and a depth of mooring parameter b/T. In the next section we present results for a wide range of T/λ (from 0 to 2 computed and plotted in increments of 0.01) and for various relevant combinations of the remaining three parameters.

Output data

Regarding the output data we choose to calculate and present graphically three output quantities which are significant for engineering applications and which characterize the so-called breakwater performance. These three quantities are: T_c, the transmission coefficient (equation 14); f, the non-dimensional mooring force given by:

$$f=F/(\rho g r_0\lambda) \tag{18}$$

where F is given in equation (12); and d, the non-dimensional horizontal displacement of the point O:

$$d=|v|/(\sigma r_0) \tag{19}$$

The influence of the input data on the output data is presented in detail in the following section.

RESULTS AND DISCUSSION

The results are presented in three groups of figures regarding the various influences on the performance of the breakwater. The first group gives an indication about the influence of the mass of the breakwater; the second shows the influence of the stiffness of mooring; and the last focuses on the influence of the location of the mooring point.

The influence of the mass parameter

In order to check the influence of the mass on the performance of the breakwater we set $K=0$ and give the mass parameter the values $\delta/\lambda=0$, 0.01, and 0.1.

Figure 2 shows the variation of the transmission coefficient T_c as a function of the draft T/λ for the above mentioned three selected values of δ/λ. An additional curve, representing a fixed plate is also presented in this Figure. For a fixed plate the transmission coefficient is given by[6]:*

$$T_c=K_1/((\pi I_1)^2+K_1^2)^{1/2} \tag{20}$$

There is almost no doubt that additional curves, for any numerical value of δ/λ, will pass between the curve for $\delta/\lambda=0$ and the curve for a fixed plate (which seems to fit the hypothetical case of $\delta/\lambda=\infty$). From Fig. 2 we see that the transmission coefficient for the case $\delta/\lambda=0.01$ is almost the same as that for a weightless breakwater, and that the curve for the case $\delta/\lambda=0.1$, (which seems to be a reasonable upper limit to what one still may call a 'thin' plate) has a trend similar to the curve for $\delta/\lambda=0$ but with smaller numerical values. For example, the required draft to obtain $T_c=0.5$ is equal to $T/\lambda\simeq0.70$, 0.57 and 0.13 for the cases $\delta/\lambda=0$, 0.1 and the fixed plate, respectively. Thus it is concluded that, in order to achieve significant influence, very big masses are needed and, when considering a breakwater with a small mass, it is recommended to use the minimal mass required by structural considerations.

Figure 3 shows the horizontal displacement d for a free breakwater ($K=0$). In the case of a weightless breakwater ($\delta/\lambda=0$) the displacement decreases from $d\simeq1$ for T/λ

* It is pointed out that an error has crept in equation (17.5) of ref 6, as well as in other places in the same section, namely the π appearing under the square-root sign should be replaced by π^2.

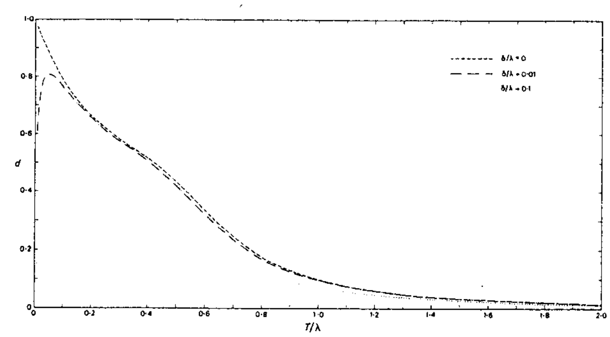

Figure 3. The influence of the mass parameter on the horizontal displacement (K = 0)

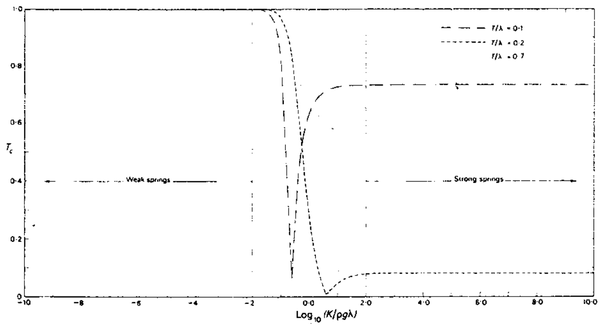

Figure 4. The transmission coefficient, for T/λ = 0.1, 0.2 *and* 0.7, *as a function of the mooring stiffness parameter*

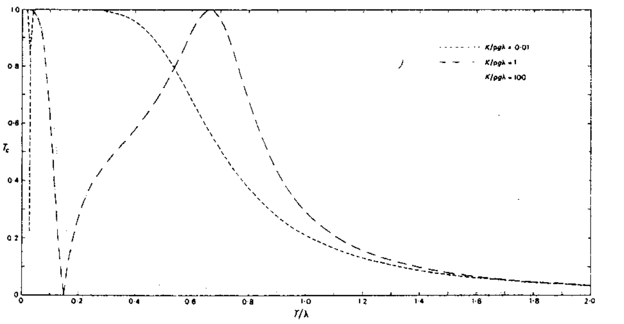

Figure 5. The influence of the mooring stiffness parameter on the transmission coefficient ($\delta = 0$, b = 0)

$= 0^+$ to $d < 0.02$ for $T/\lambda = 2$. The curve for $\delta/\lambda = 0.1$ indicates significantly smaller oscillation than those for $\delta/\lambda = 0$, especially for small drafts.

The influence of the mooring stiffness parameter

In order to study only the effects of the stiffness parameter the mass parameter is removed by setting $\delta = 0$. The point O (intersection of breakwater with water surface) is chosen arbitrarily as the mooring point, so that $b = 0$. In all three Figures, (Fig. 5 for T_c, Fig. 6 for f and Fig. 7 for d), three different springs are considered: a weak spring for which $K/\rho g\lambda = 0.01$, a medium spring, $K/\rho g\lambda = 1$, and a strong one, $K/\rho g\lambda = 100$. In order to give an indication about the origin of the terminology (weak spring or strong spring) we show in Fig. 4 the variation of the transmission coefficient, for the different drafts $T/\lambda = 0.1$, 0.2 and 0.7, as a function of $K/\rho g\lambda$. It can be seen in this Figure that for all values of $K/\rho g\lambda$ above 100 there is no influence of the stiffness on T_c, hence all springs having $K/\rho g\lambda \geqslant 100$ are considered as 'strong springs'. Similarly it can be seen that there is no effect on T_c for $K/\rho g\lambda$ below 0.01 which is considered as the limit of 'weak springs'.

The transmission coefficient for the weak spring (Fig. 5) is very similar to that of a free breakwater (Fig. 2) except in the neighbourhood of $T/\lambda \simeq 0.03$ where a sharp decrease in T_c is noticed. This decrease in T_c is involved with increase of the mooring force (which is elsewhere very small for this spring, see Fig. 6), and an 'explosion' in the horizontal displacement (Fig. 7). The numerical calculations near $T/\lambda = 0.03$ employed T/λ increments of 4×10^{-4} and resulted in a maximum displacement $d = 16$. There is almost no doubt that this result indicates a resonance phenomenon and higher values of d should be expected when using smaller increments of T/λ in the calculation. The behaviour of the breakwater with a strong spring is generally similar to that of a rigid plate, for which the non-dimensional horizontal force is given[6], by

$$f = 2TS_1/\lambda(\pi^2 I_1^2 + K_1^2)^{1/2} \tag{21}$$

The most interesting results are obtained for the medium spring ($K/\rho g\lambda = 1$). In this case the transmission coefficient (Fig. 5) is smaller than that for a rigid plate for $T/\lambda < 0.19$ (for $T/\lambda \simeq 0.15$, $T_c = 0$). On the other hand, for $T/\lambda \geqslant 0.15$ the transmission coefficient of the medium spring increases and reaches $T_c = 1$ for $T/\lambda \simeq 0.67$. Generally speaking, for $T/\lambda > 0.53$ the performance of a breakwater with medium mooring is worse than that without any mooring.

Note that the phenomenon, in which T_c reaches the values of 0 and 1 (for the medium spring) resembles the results for a configuration of two thin vertical barriers obtained by Srokosz and Evans[5]. To conclude, the various and sometimes surprising phenomena, caused by the introduction of a mooring system, call for caution in breakwater design.

The influence of the depth of mooring parameter

Here we set $\delta = 0$, $K/\rho g\lambda = 1$ and check the results for three different points of mooring. Namely, mooring at the water surface, $b/T = 0$; at the middle point of the breakwater $b/T = 0.5$, and at the lower edge $b/T = 1$. Generally speaking, the differences between the various mooring depths are insignificant, regarding the transmission coefficient (Fig. 8) and the mooring forces (Fig. 9). However, from Fig. 10 it is seen that mooring at the lower edge of the plate allows larger displacements (significantly larger in

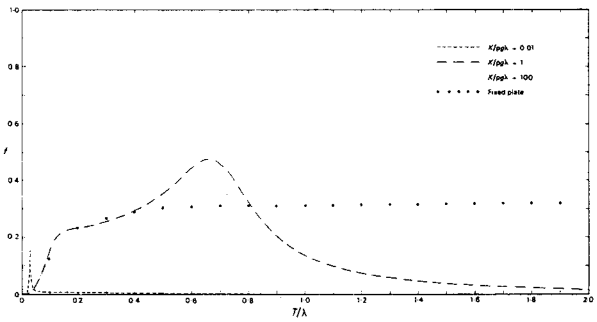

Figure 6. The influence of the mooring stiffness parameter on the mooring force ($\delta = 0$, b = 0)

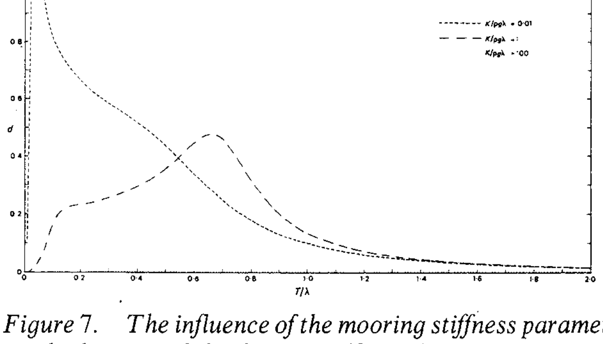

Figure 7. The influence of the mooring stiffness parameter on the horizontal displacement ($\delta = 0$, b = 0)

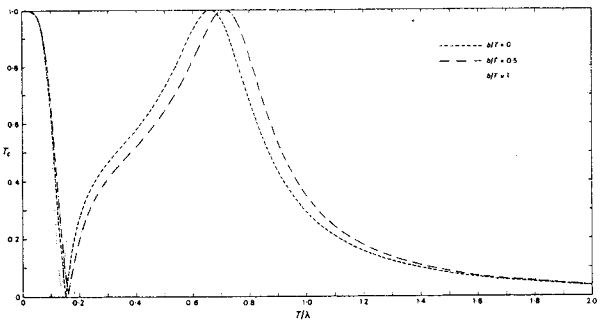

Figure 8. The influence of the depth of mooring on the transmission coefficient ($\delta = 0$, $K/\rho g\lambda = 1$)

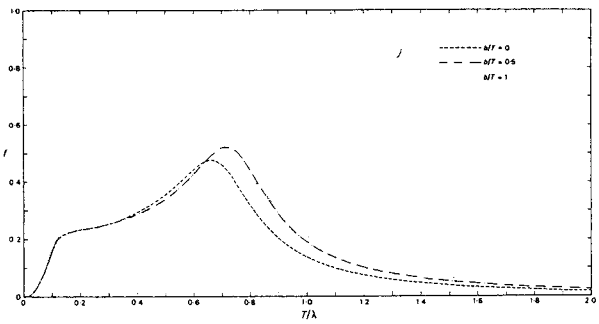

Figure 9. The influence of the depth of mooring on the mooring force ($\delta = 0$, $K/\rho g\lambda = 1$)

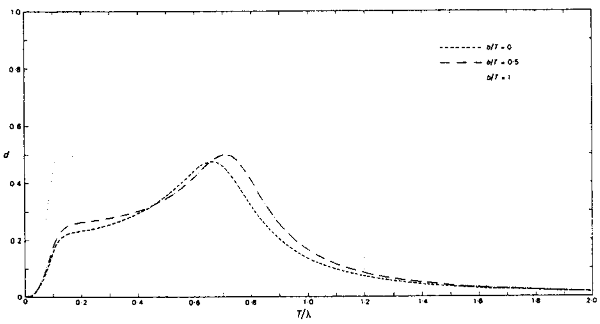

Figure 10. The influence of the depth of mooring on the horizontal displacement, ($\delta = 0$, $K/\rho g\lambda = 1$)

the region where $T_c \simeq 0$) than the other two alternatives, as should be expected.

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APPENDIX

The expressions for the coefficients, appearing in equations (6) and (7) are as follows:

$$\lambda_{22} = 4\rho\sigma T^2 S_1^2/[\pi^2 I_1^2(\mu) + K_1^2(\mu)]$$

$$\lambda_{44} = 4\rho\sigma T^4 (S_1 - \pi/4)^2/\{\mu^2[\pi^2 I_1^2(\mu) + K_1^2(\mu)]\}$$

$$\lambda_{24} = \lambda_{42} = 4\rho\sigma T^3 S_1(S_1 - \pi/4)/\{\mu[\pi^2 I_1^2(\mu) + K_1^2(\mu)]\}$$

$$\mu_{22} = (4/\pi)\rho T^2\{0.5 - S_0/\mu + S_0^{-1}/\mu^2 - S_1(\Gamma/\mu)/[\pi^2 I_1^2(\mu) + K_1^2(\mu)]\}$$

$$\mu_{24} = (4/\pi)\rho T^3\{\pi/12 + 1/2\mu - S_0/\mu^2 + S_0^{-1}/\mu^3 - [(S_1\Gamma - \pi\Gamma_0/4)/\mu^2]/[\pi^2 I_1^2(\mu) + K_1^2(\mu)]\}$$

$$\mu_{42} = (4/\pi)\rho T^3\{\pi/12 + 1/2\mu - S_0/\mu^2 + S_0^{-1}/\mu^3 - [(S_1 - \pi/4)\Gamma/\mu^2]/\pi^2 I_1^2(\mu) + K_1^2(\mu)]\}$$

$$\mu_{44} = (4/\pi)\rho T^4\{1/2\mu^2 - \pi/12\mu - \pi^2/64 - (1/\mu^3 + \pi/4\mu^2)S_0 + S_0^{-1}/\mu^4 - (S_1 - \pi/4)[(\Gamma/\mu - \mu\gamma_2\pi/4)/\mu^2]/[\pi^2 I_1^2(\mu) + K_1^2(\mu)]\}$$

In the above formulae Γ, Γ_0 and γ_2 are given by:

$$\Gamma = \gamma_1 - \mu\gamma_2 - 0.5\pi K_1(\mu)$$

$$\gamma_1 = \pi^2 I_0^{-1}(\mu) I_1(\mu) - K_0^{-1}(\mu) K_1(\mu)$$

$$\gamma_2 = \pi^2 I_0(\mu) I_1(\mu) - K_0(\mu) K_1(\mu)$$

$$\Gamma_0 = \mu^2 S_1 \gamma_2 - \mu S_0[\pi^2 I_1^2(\mu) + K_1^2(\mu)]$$

where $S_0 = 0.5\pi[I_0(\mu) + L_0(\mu)]$, L_0 is a Struve modified function of order zero, K_0, K_1, I_0, I_1 are Bessel functions, and

$$S_0^{-1} = \int_0^\mu S_0(\tau)\mathrm{d}\tau; \qquad I_0^{-1} = \int_0^\mu I_0(\tau)\mathrm{d}\tau;$$

$$K_0^{-1} = \int_0^\mu K_0(\tau)\mathrm{d}\tau$$

The coefficients in equations (13) and (14) are:

$$B_2 = 2TS_1/[\pi I_1(\mu) - jK_1(\mu)]$$

$$B_4 = 2T^2(S_1 - \pi/4)/\{\mu[\pi I_1(\mu) - jK_1(\mu)]\}$$

$$B_7 = \pi\sigma r_0 T I_1(\mu)/\{\mu[\pi I_1(\mu) - jK_1(\mu)]\}$$

ERRATUM

'A simple mathematical model of a floating breakwater', by M. Stiassnie, *Applied Ocean Research*, 1980, **2**, 107–111:

The author regrets overlooking an algebraic error in one of Haskind's formulae. The term:

$$(\rho T^4/\mu^2)\cdot(\pi\mu^2/8+\mu+\pi/2)$$

should be added to the expression for μ_{44} in his equation (2.18) as well as in the Appendix of the paper.

The error was detected as a result of a poor fit which was obtained while comparing the mathematical model with results of some preliminary wave flume experiments.

All the other expressions for the damping and added mass coefficients as well as the new expression for μ_{44} yield numerical values which are the same as those given in tabulated forms by Kotik (1963), Porter (1965) and Mei (1976) (all of them were published in the *Journal of Ship Research*).

Unfortunately, the correction for μ_{44} results in some quantitative changes in the Figures of the paper.

Readers who are interested in the corrected set of Figures are advised to contact the author directly.