Extreme Values of Breaker Direction and Longshore Current

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Introduction

Considering a straight shoreline and parallel bottom contours, the refraction of a periodic wave is described by Snell's law as

$$\frac{\sin \alpha}{\sin \alpha_0} = \frac{L}{L_0} \quad (1)$$

in which α = the angle between wave front and bottom contour; L = the wave length; and subscript 0 refers to deep water conditions. According to the linear theory, L is related to the local depth, d, by

$$\frac{L}{L_0} = \tanh \frac{2\pi d}{L} \quad (2)$$

Assuming conservation of wave energy flux between orthogonals, the variation in wave height H (due to both refraction and shoaling) is described by

$$\frac{H}{H_0} = \left(\frac{L}{L_0}\right)^{-1/2} \left[1 + \frac{\left(4\pi \frac{d}{L}\right)}{\sinh\left(4\pi \frac{d}{L}\right)}\right]^{-1/2} \left(\frac{1 - \sin^2 \alpha_0}{1 - \sin^2 \alpha}\right)^{1/4} \quad (3)$$

Eqs. 1, 2, and 3 may be found in almost any book on water waves, e.g., Le Méhauté (3) and we assume them valid, at least approximately, up to the first breakerline.

The criterion selected for wave breaking is the usually adopted condition

$$\frac{H_b}{d_b} = 0.8 \quad (4)$$

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in which subindex b denotes breaking. It should be noted that Eq. 4 sets a practical limit for H_b before an orthogonal crossing singularity is encountered along the rays. An examination of the numerical value in the right-hand side of Eq. 4 can be found in Galvin (2).

Eqs. 1–4 can be solved in order to provide the values H_b, L_b, d_b, and α_b for given deep water conditions H_0, L_0, and α_0. After doing that, an estimate for the peak U of the longshore current distribution, for the more restrictive case of a plane bed of slope s, is given by

$$U = \left(28 \frac{s}{d_b}\right) g^{1/2} H_b^{3/2} \sin(2\alpha_b) \quad (5)$$

in which g = the gravitational acceleration. Eq. 5 was proposed by Balsillie (1) following Longuet-Higgins (4).

While observing tables of numerical results for Eqs. 1–5, the writers noticed that extreme values for α_b and U occurred consistently at α_0 values a little higher than 60° and a little smaller than 60°, respectively, with almost no dependence on the other two deep water parameters, H_0 and L_0. It is the aim of this note to confirm analytically the values of α_0 at which the previously mentioned extreme conditions occur.

In the framework of linearized wave theory, using the classical wave breaking criterion and longshore current formula, the following two extreme conditions are analytically derived: (1) The maximum angle between the breakerline and the shoreline occurs when the angle between wave fronts in deep water and the shoreline is about 66°; and (2) maximum longshore current velocities are obtained when the previously mentioned (deep water) angle is about 58°.

Maximum Breakerline to Shoreline Angle

Elimination of H_b, L_b, and d_b in Eqs. 1–4, yields the following rather involved expression for α_b as dependent upon deep water parameters. Thus

$$\left(\frac{a_b}{a_0}\right)^{3/2} \tanh^{-1}\left(\frac{a_b}{a_0}\right) = \delta \left(\frac{1 - a_0^2}{1 - a_b^2}\right)^{1/4} \Bigg/ \left\{1 + 2\tanh^{-1}\left(\frac{a_b}{a_0}\right) \Bigg/ \sinh\left[2\tanh^{-1}\left(\frac{a_b}{a_0}\right)\right]\right\}^{1/2} \quad (6)$$

in which $a_b = \sin \alpha_b$; $a_0 = \sin \alpha_0$; and $\delta = (2\pi/0.8)\, H_0/L_0$. The parameter δ is a measure of the wave steepness in deep water, which is assumed to be small, i.e., $\delta = 0\,(1)$.

Eq. 6 indicates that small values of δ imply small values of a_b/a_0. The asymptotic expansion of Eq. 6 for small values of a_b/a_0 is found to yield

$$a_b = 2^{-1/5} a_0 (1 - a_0^2)^{1/10} \delta^{2/5} [1 - b_0 \delta^{4/5} + 0(\delta^{8/5})] \quad (7a)$$

in which $$b_0 = (1/30)\, 2^{-2/5} (2 - 3a_0^2)(1 - a_0^2)^{1/5} \quad (7b)$$

Eq. 7a indicates that, to accuracy $0(\delta^{6/5})$, a_b and consequently α_b are proportional to $\delta^{2/5}$.

In order to calculate the value of a_0 making α_b a maximum (we denote this value with A_0), the derivative with respect to a_0 of Eq. 7a is equated to zero. This results in

$$A_0 = \left(\frac{5}{6}\right)^{1/2}\left[1 + \frac{\delta^{4/5}}{30 \times 3^{6/5} \times 2^{3/5}} + 0(\delta^{8/5})\right]$$

$$= 0.9129\ [1 + 0.0059\delta^{4/5} + 0(\delta^{8/5})] \quad (8)$$

Eq. 8 indicates an extremely weak dependence of A_0 upon δ, so that for almost all practical purposes, the maximum value of α_b is expected to appear at

$$\alpha_0 = \sin^{-1} A_0 \approx \sin^{-1}(0.9129) \approx 66° \quad (9)$$

The extreme value of α_b itself is obtained by substituting Eq. 8 into Eq. 7a, or when neglecting terms of $0(\delta^{6/5})$, Eq. 9 into Eq. 7a. The latter substitution yields

$$\alpha_{b\max} = \sin^{-1}(0.6643\,\delta^{2/5}) \quad (10)$$

Maximum Longshore Current (Plane Bed of Slope s)

Expanding Eq. 5 for small δ and using Eq. 7a, the following expression for the peak velocity of the longshore current distribution is obtained

$$U = C_0 s (0.8)^{3/2}\, 2^{3/5}\, 28\, a_0(1 - a_0^2)^{1/5}\delta^{4/5}\,[1 + e_0\delta^{4/5} + 0(\delta^{8/5})] \quad (11a)$$

in which C_0 = the wave celerity in deep water and

$$e_0 = \frac{1}{30} 2^{-2/5}(1 - 9a_0^2)(1 - a_0^2)^{1/5} \quad (11b)$$

In order to calculate the value of a_0 making U a maximum (we denote this value with $\underline{A}_0$), the derivative with respect to a_0 of Eq. 11a is equated to zero. After several manipulations, one eventually obtains

$$\underline{A}_0 = \left(\frac{5}{7}\right)^{1/2}\left[1 - \frac{(13 \times 2^{9/5})}{(30 \times 7^{11/5})}\delta^{4/5} + 0(\delta^{8/5})\right]$$

$$= 0.8452\ [1 - 0.0209\delta^{4/5} + 0(\delta^{8/5})] \quad (12)$$

Although stronger than in Eq. 8, the dependence of $\underline{A}_0$ upon δ is still weak. The numerical value of α_0 corresponding to $\underline{A}_0$ is, for practical purposes

$$\alpha_0 = \sin^{-1}(\underline{A}_0) \approx \sin^{-1}(0.8452) \approx 58° \quad (13)$$

Substituting this result in expression Eq. 11a, one obtains

$$U_{\max} = 19.98\, sC_0\delta^{4/5} \quad (14)$$

Remarks and Conclusion

Since the definition of δ includes the numerical value 0.8 selected for the breaking criterion, Eq. 4, the results hold for any other such value as long as δ remains small. Only in the expression for the peak velocity, Eq. 11a, the value appears additionally, as a 3/2 power factor influencing the numerical coefficient in Eq. 14. In fact, the tables of numerical results that inspired this note used the more general breaking criterion known as Miche's formula (see Ref. 2, p. 419). Furthermore, the numerically observed trend of the extreme values indicates the validity of our asymptotic results, even for values of δ as large as 0.75, which covers almost all practical cases.

The analytical derivation of both extreme conditions, although unpretentious in itself, yields a theoretical background for a perhaps known fact. The present results confirm the trends observed in tables of breaker parameters and in physical models in use at the Coastal and Marine Engineering Research Institute. The results may be helpful for a quick estimation of extreme longshore current and sediment transport conditions as well as for the design of tests in physical models where those conditions are relevant.

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Appendix.—References

1. Balsillie, J. H., "Surf Observations and Longshore Current Prediction," *C.E.R.C. Technical Memorandum No. 58,* U.S. Army Corps of Engineers Coastal Engineering Research Center, Ft. Belvoir, Va., 1975, 39 p.
2. Galvin, C. J., "Wave Breaking in Shallow Water," *Waves on Beaches,* R. E. Meyer, ed., Academic Press, New York, N.Y., 1972, pp. 413–456.
3. Le Méhauté, B., *An Introduction to Hydrodynamics and Water Waves,* Springer-Verlag, New York, N.Y., 1976, 315 p.
4. Longuet-Higgins, M. S., "Recent Progress in the Study of Longshore Current," *Waves on Beaches,* R. E. Meyer, ed., Academic Press, New York, N.Y., 1972, pp. 203–248.