Viscous Parallel Flow Along Cylindrical Surfaces Described by Algebraic Polynomials

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ABSTRACT

Steady laminar parallel flow along fixed cylindrical surfaces or within cylinders or prisms, obeys the Navier-Stokes equations and is described by a linear inhomogeneous partial differential equation of the elliptic (Poisson) type. A survey is made of the various boundaries which are described by algebraic polynomials of order 2, 3, and 4, and recursion formulas are given for polynomials of a higher order. For open profiles, e.g. between two intersecting planes, or within hyperbolic cylinders, more than one flow can be obtained for the same longitudinal pressure gradient. The discussion is extended to unsteady flow, which obeys a linear inhomogeneous parabolic equation. Applications are mentioned, e.g. in boundary layer theory; explanations of the so-called irreducible (ineffective) porosity in porous media; torsion of shafts with algebraic profiles, etc.

NOTATION

A, a, b	— coefficients	x, y, z	— cartesian coordinates
D	— discriminant	$x + iy$	— complex number
F, f	— function	Z	— elevation
$h, k, m, n,$	— constants	α, β, γ	— coefficients
N	— order of polynomial	ν	— kinematic viscosity
P	— $p/\rho + gZ$	ρ	— density
p	— pressure	$\phi = 1/\nu$	— fluidity
R	— real part of complex number	ω	— harmonic function
t	— time	$w_t = \partial w/\partial t$	
V	— velocity vector	$w_{xx} = \partial^2 w/\partial x^2,$	
$u, v, w,$	— its cartesian components	$w_{yy} = \partial^2 w/\partial y^2$	
w_0	— maximum velocity		

INTRODUCTION

Viscous flow of a constant-density liquid obeys the linear continuity equation

$$\operatorname{div} V = 0 \tag{1}$$

and the non-linear Navier-Stokes equations

$$V_t + \operatorname{curl} V \times V + \operatorname{grad}(V^2/2) = -\operatorname{grad} P + \nu\nabla^2 V \tag{2}$$

where

$$P = p/\rho + gZ \tag{3}$$

is proportional to the piezometric head $(Z + p/g\rho)$.

In the special case of flows parallel to the z-axis of a cartesian coordinate system (x, y, z), the velocity vector $V\ (u, v, w)$ has only one non-vanishing component (w), which at a given instant is constant along the streamline. Equations (2) reduce to a linear inhomogeneous second order partial differential equation of the parabolic type

$$w_t = gJ + \nu(w_{xx} + w_{yy});\ gJ = -P_z \tag{4}$$

J is the piezometric gradient, which in our case of constant velocity is either constant or a function of

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time (t) only. When it is constant, we shall consider only the case $J > 0$, i.e. of piezometric head decreasing in the direction of flow.

In stationary flow Eq. (4) becomes a linear inhomogeneous second order partial differential equation of the elliptic type, the so-called Poisson equation

$$w_{xx} + w_{yy} + 2j = 0 \,; \quad j = gJ/2\nu \tag{5}$$

Its general solution is

$$w = -jy^2 + \omega \tag{6}$$

ω is the solution of the harmonic (Laplace) equation

$$\omega_{xx} + \omega_{yy} = 0 \tag{7}$$

We assume that the flow boundary is a fixed cylindrical wall of generatrices parallel to the z-axis and of equation $f(x,y) = 0$. By the non-slip condition the boundary condition at that surface is

$$w = 0 \quad \text{at} \quad f(x,y) = 0 \tag{8}$$

When the flow occurs within a cylinder whose cross-section is a closed curve, Eqs. (7), (8) have a unique solution. In an unbounded domain this is not always so, as will be shown later.

GENERAL SOLUTION

Berker (1963) gives the general solution and some particular solutions of Eq. (5).

Equation (7) is satisfied by the real (R) or imaginary part of any analytic function of the complex variable $(x + iy)$

$$R\zeta = RF(x+iy) \tag{9}$$

Among the numerous functions $f(x,y)$ obeying Eq. (7) a special case is that of a power series in $(x + iy)$

$$F(x+iy) = \sum_{k=0}^{N} (b_k + ib'_k)(x+iy)^k \tag{10}$$

b_k, b'_k are real numbers. By Eq. (6)

$$w = \sum_{m,n=0}^{m+n=k\leq N} A_{mn}x^m y^n - jy^2 \tag{11}$$

As we consider only flow in the direction of decreasing head (or pressure), $j > 0$

$$w/j = \sum_{m,n=0}^{m+n=k\leq N} a_{mn}x^m y^n - y^2 = f(x,y) \tag{12}$$

$$\left.\begin{aligned} a_{mn} = A_{mn}/j &= b_k\binom{k}{n} \quad (n = k-m = 4n_1) \\ &= -b'_k\binom{k}{n} \quad (n = k-m = 4n_1+1) \\ &= -b_k\binom{k}{n} \quad (n = k-m = 4n_1+2) \\ &= b'_k\binom{k}{n} \quad (n = k-m = 4n_1+3) \\ \binom{k}{n} &= k!/m!n! \quad (n = k-m) \end{aligned}\right\} \tag{13}$$

By the non-slip condition (8), the velocity w vanishes along the cylindrical surface of the algebraic equation of order N

$$f(x,y) = \sum_{m,n=0}^{m+n=k\leq N} a_{mn}x^m y^n - y^2 = 0 \tag{14}$$

Another way of computing the coefficients a_{mn} is to introduce Eq. (12) into (5) and equate to 0 all the coefficients of the terms in $x^m y^n$. We obtain

$$\left.\begin{aligned} a_{20} + a_{02} + 1 = 0 \\ (m+2)(m+1)a_{m+2,n} + (n+2)(n+1)a_{m,n+2} = 0; \\ m+n > 2 \end{aligned}\right\} \tag{15}$$

Developing w into polynomials of the fifth order

$$\begin{aligned} w/j = f(x,y) &= a_{00} + [a_{10}x + a_{01}y] + [a_{20}x^2 + \\ &a_{11}xy - (1 + a_{20})y^2] + [a_{30}(x^3 - 3xy^2) + \\ &+ a_{03}(y^3 - 3x^2y)] + [a_{40}(x^4 - 6x^2y^2 + y^4) + \\ &+ a_{31}(x^3y - xy^3)] + [a_{50}(x^5 - 10x^3y^2 + 5xy^4) + \\ &+ a_{05}(5x^4y - 10x^2y^3 + y^5)] + \cdots \end{aligned} \tag{16}$$

The coefficients a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{30}, a_{03}, a_{40}, a_{31}, a_{50}, $a_{05}, \cdots$ are arbitrary. Any other coefficient a_{mn} is of the form (13) with b_k, b_k' arbitrary. Equation (16) is expressed as the sum of polynomials of order $1, 2, 3, 4, \cdots$

$$w/j = f(x,y) = a_{00} + f^{I} + f^{II} + f^{III} + f^{IV} + \cdots \tag{17}$$

The equation of an isovel (or isotach, line of equal velocity) is $w = w_1$. At the boundary, by Eq. (8)

$$w = f(x, y) = 0$$

This is the zero isovel. We may take any isovel $w = w_1$ as a new boundary, if the new velocity is

$$w' = f(x, y) - w_1 \tag{18}$$

This is also a possible flow, as it obeys Eq. (5).

In what follows we shall survey the polynomial solutions (16) of successive orders 2, 3, 4.

POLYNOMIALS OF ORDER 2

The quadratic equation

$$f(x, y) = a_{00} + f^I + f^{II} = a_{00} + (a_{10}x + a_{01}y) + \\ + [a_{20}x^2 + a_{11}xy - (a_{20} + 1)y^2] \tag{19}$$

has the discriminant

$$D = a_{11}^2 + 4a_{20}(a_{20} + 1) \tag{20}$$

For $D < 0$, i.e., when $-1 < a_{20} < 0$, $f(x, y) = 0$ gives an *ellipse* of semi-axes (a, b). After displacement and rotation of the axes, its canonical equation is

$$x^2/a^2 + y^2/b^2 = 1 \tag{21}$$

The velocity $w(x, y)$ is

$$w/w_0 = 1 - x^2/a^2 - y^2/b^2; \; w_0 = j/(1/a^2 + 1/b^2) \tag{22}$$

The isovels are coaxial similar ellipses. The velocity distribution curve along the x-axis is a parabola concave downwards with a maximum $w_0 = w(0, 0)$ at the center (Fig. 1). No such flow is possible outside the ellipse, as there $w < 0$ for $j > 0$.

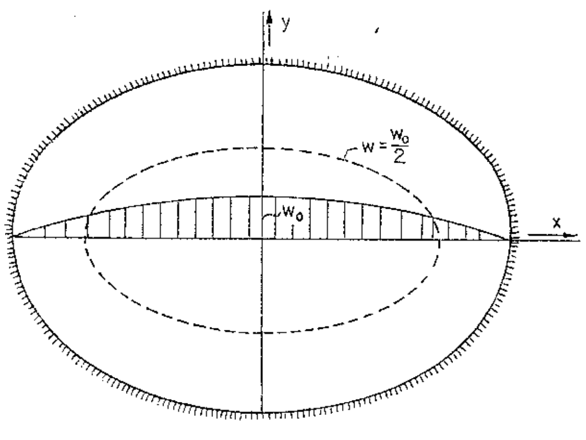

Fig. 1. Ellipse.

In the special case of *circular* boundary of radius a

$$f(x, y) \equiv x^2 + y^2 - a^2 = 0$$

$$w/w_0 = 1 - (x^2 + y^2)/a^2; \quad w_0 = ja^2/2$$

For $D > 0$ we get a hyperbola whose canonical form is

$$x^2/a^2 - y^2/b^2 = 1 \tag{23}$$

For $a < b$, i.e. $-1 < a_{20} < 0$ the velocity is

$$w/w_0 = 1 - x^2/a^2 + y^2/b^2; \; w_0 = j/(1/a^2 - 1/b^2) \tag{24}$$

For $a > b$, i.e. $a_{20} > 0$, it is

$$w/w_0 = x^2/a^2 - y^2/b^2 - 1; \; w_0 = j/(1/b^2 - 1/a^2) \tag{25}$$

Equation (24) represents flow between the two branches of the hyperbola (23) of asymptotes $\pm y/x = b/a > 1$ (Fig. 2). The isovels are hyperbolas with the same asymptotes. Along the asymptotes $w = w_0$; near the y-axis $w > w_0$; near the hyperbolas $w < w_0$. The velocity curve along the x-axis is a parabola concave downwards; and along the y-axis a parabola convex downwards.

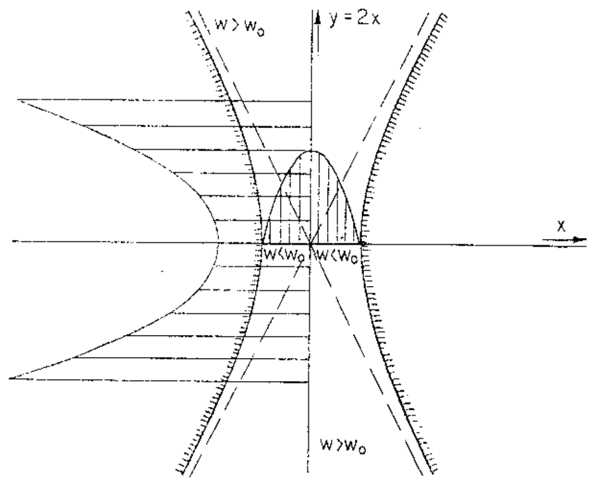

Fig. 2. Hyperbola.

Equation (25) represents flow inside the hyperbola (23) of asymptotes $\pm y/x = b/a < 1$ (Fig. 3). The isovels are hyperbolas with the same asymptotes. Along the x-axis $w(x)$ is a parabola convex downwards.

Interverting x and y, a and b, Eq. (24) becomes

$$w/w_0 = 1 + x^2/a^2 - y^2/b^2; \; w_0 = j/(1/b^2 - 1/a^2); \quad b < a \tag{26}$$

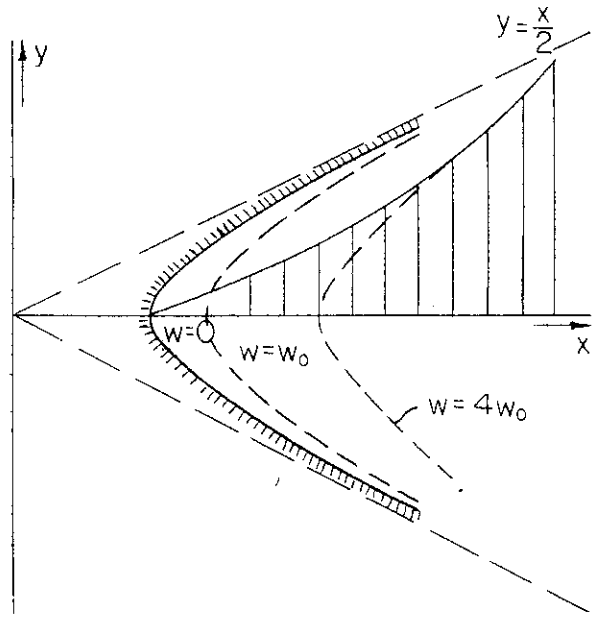

Fig. 3. Hyperbola.

Flow occurs inside the hyperbolas of asymptotes $\pm\, y/x = b/a < 1$.

Interverting x and y, a and b, Eq. (25) becomes

$$w/w_0 = y^2/b^2 - x^2/a^2 - 1; \; w_0 = j/(1/a^2 - 1/b^2); \quad b > a \quad (27)$$

Flow occurs between hyperbolas of asymptotes $\pm\, y/x = b/a > 1$.

A special case is flow between two intersecting planes $y = 0$, $y = mx$, enclosing an angle $\theta = \tan^{-1}m$ (Fig. 4). Then a_{00}, a_{01}, a_{10}, $a_{20} = 0$, and

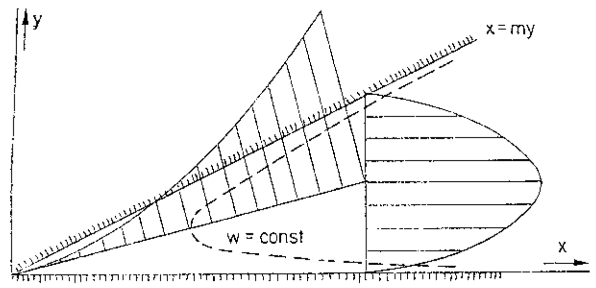

Fig. 4. Angle.

$$w/j = y(mx - y) \quad (28)$$

The isovels are hyperbolas asymptotic to the two boundaries. The curve $w(y)$ at a given x is a parabola concave towards the base. Along the bisector it is a parabola convex downwards.

When the two planes are parallel ($\theta = 0$) of distance h, we get the Poiseuille flow

$$w/j = y(h - y) \quad (29)$$

$w(y)$ is a parabola concave to the base.

For $D = 0$ we get flow inside a *parabolic* cylinder of the canonical equation and velocity

$$x = y^2/k; \quad w = j(kx - y^2); \quad a_{20}, a_{11} = 0 \quad (30)$$

The isovels (Fig. 5) are parallel parabolas. The curve $w(x)$ at a given x is a parabola concave towards the base.

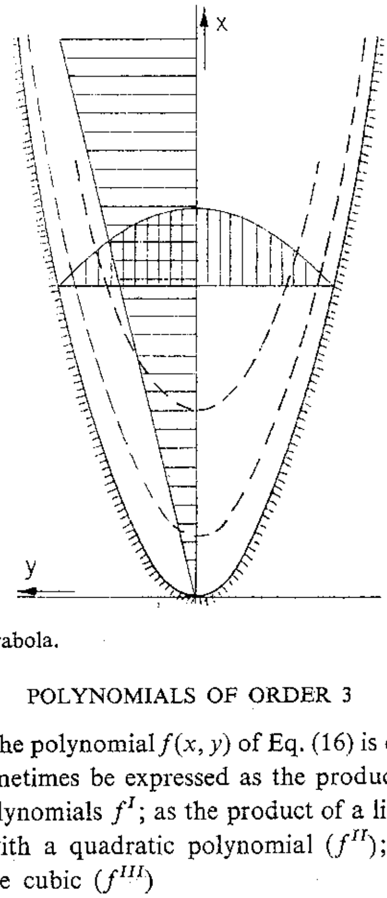

Fig. 5. Parabola.

POLYNOMIALS OF ORDER 3

When the polynomial $f(x, y)$ of Eq. (16) is of order 3, it can sometimes be expressed as the product of three linear polynomials f^I; as the product of a linear polynomial with a quadratic polynomial (f^{II}); or as an irreducible cubic (f^{III})

$$\left.\begin{aligned} f &= f_1^I f_2^I f_3^I \\ f &= f^I f^{II} \\ f &= f^{III} \end{aligned}\right\} \quad (31)$$

Case $f = f_1^I f_2^I f_3^I$

By an adequate choice of axes we may write

$$w/j = Ky(ax - y)(b - y - mx) \qquad (32)$$

The only solution is that of an equilatreal triangle of side b

$$m = a = \sqrt{3}; \qquad k = 1/b \qquad (33)$$

The isovels $w = w_1$ are closed curves of equation

$$w/w_0 = (54/b^3) \cdot y(\sqrt{3}x - y)(b - y - \sqrt{3}x) = w_1/w_0 \qquad (34)$$

The curve $w(y)$ in the section through the centre and a vertex (Fig. 6) has a maximum $w_0 = jb^2/54$ at the lower third and an inflexion point at the upper third, with $w/w_0 = 1/2$. At the vertex the curve is tangent to the base.

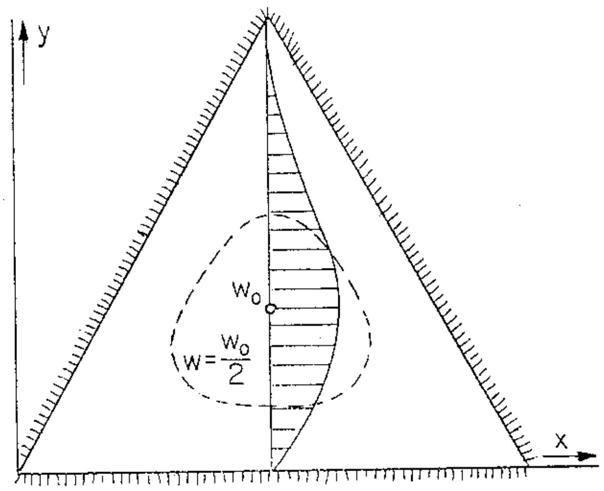

Fig. 6. Equilateral triangle.

By Eq. (18) flow within a cylinder of the form of an isovel (34) is represented by

$$w/w_0 = (54/b^3).\, y(\sqrt{3x} - y)(b - y - \sqrt{3x}) - w_1/w_0; \qquad w_1 < w_0 \qquad (35)$$

When choosing $w_1 > w_0/2$, we obtain a curve $w(y)$ through the center without inflexion.

Case $f = f^I f^{II}$

Possible solutions represent flow between a plane ($f^I = 0$) and a hyperbolic cylinder ($f^{II} = 0$) whose asymptotes are $\pm\, y/x = \sqrt{3}$

$$w/j = y[a_{01} + a_{11}x - y + a_{03}(y^2 - 3x^2)] \qquad (36)$$

Without loss of generality we may assume $a_{11} = 0$. Several subcases are possible.

$\underline{a_{03} > 3}$

$$w/ja_{03} = y[(y - \beta)^2 - 3x^2 \pm \gamma^2]$$

$$\beta = 1/2a_{03} > 0; \;\; \gamma^2 = (4a_{03}a_{01} - 1)/4a_{03}^2 \qquad (37)$$

$$a_{01} > 1/4a_{03}$$

The only possible flow is in the upper zone between the horizontal $y = 0$ and the two branches of the hyperbola (Fig. 7a)

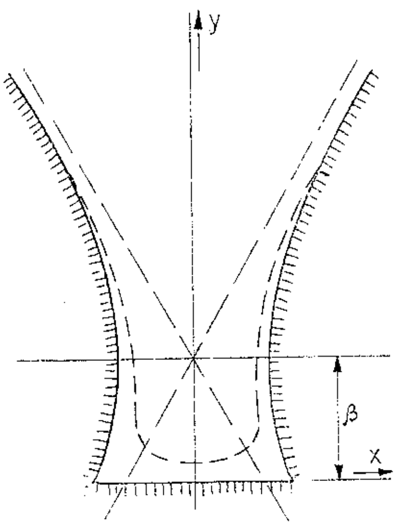

Fig. 7a.

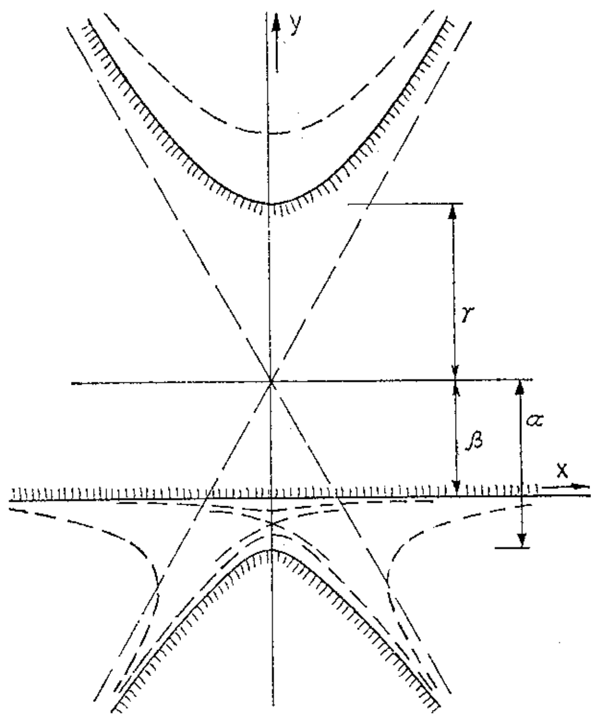

Fig. 7b.

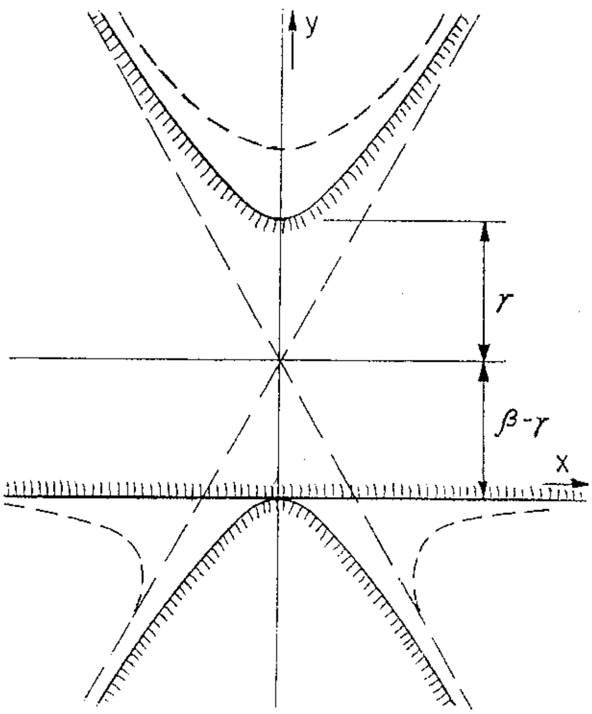

Fig. 7c.

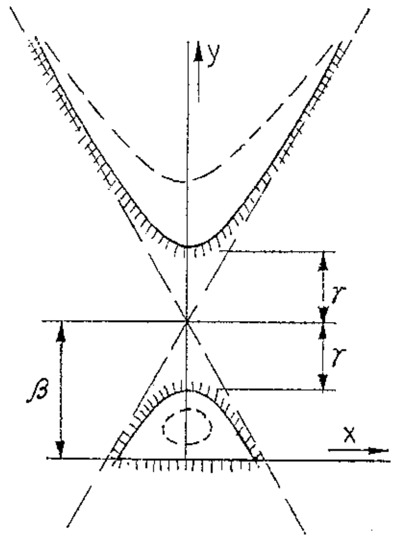

Fig. 7d.

Fig. 7. Hyperbola and straight line.

$$(y-\beta)^2 - 3x^2 + \gamma^2 = 0;$$
$$w/ja_{03} = y[(y-\beta)^2 - 3x^2 + \gamma^2] \qquad (38)$$

$a_{01} = 1/4a_{03} = \beta/2;\ \gamma = 0$

We get again flow within the equilateral triangular prism (34).

$a_{01} < 1/4a_{03};\ \gamma^2 = (1 - 4a_{03}a_{01})/4a_{03}^2$

Flow is near the branches of the hyperbola

$$(y-\beta)^2 - 3x^2 - \gamma^2 = 0;$$
$$w/ja_{03} = y[(y-\beta)^2 - 3x^2 - \gamma^2] \qquad (39)$$

with the two asymptotes $\pm(y-\beta)/x = \sqrt{3}$.

$a_{01} < 0;\ \beta < \gamma$

Flow occurs inside the upper hyperbola (39); and between the horizontal $y = 0$ and the lower hyperbola (Fig. 7b). In the latter case the velocity along the vertical $x = 0$ has a maximum at

$$y_0 = -(\sqrt{3\beta^2 + \gamma^2} - 2\beta)/3$$

The isovel $w = w_0$ has a saddle point with two isovels passing through that point. The isovels $w < w_0$ are more or less parallel to the boundaries. The isovels $w > w_0$ are two branches of a near-hyperbolic form. The existence of a saddle point singularity in parallel flow is an unusual feature. It occurs often in nature, e.g. in flow of water in the interstices of a porous medium between a rounded sand grain and a larger flat grain.

$a_{01} = 0;\quad \beta = \gamma$

Flow occurs inside the upper hyperbola (39); and in the corners between the horizontal $y = 0$ and the lower hyperbola (Fig. 7c).

$a_{01} > 0;\ \beta > \gamma$

Flow occurs inside the upper open branch of (39); and in the closed section bounded by the horizontal $y = 0$ and the lower hyperbola (Fig. 7d).

Equation (39) represents, for each value of β (or a_{03}), a different solution inside the upper hyperbola for the same value of the hydraulic gradient J (or j) (when $a_{01} \leqq 0$). Equation (5) has then no unique solution, but an infinite number of solutions compatible with the same boundaries and the same hydraulic gradients.

$a_{03} = 0$

We get a polynomial of order 2.

$a_{03} < 0$

We get similar results as for $a_{03} > 0$.

Case $f = f^{III}$

Equation (35) is an example of an irreducible cubic

f_{III}. The same applies to the cubic derived from (37)

$$w = ja_{03}y[(y-\beta)^2 - 3x^2 \pm \gamma^2] - w_1 \quad (40)$$

which is identical with an isovel of the former case.

The study of third order curves led Newton to divide them into 7 groups following the number and character of their asymptotes (*Savelov*, 1961). By Eqs. (17), (16)

$$\left.\begin{aligned} f(x,y) &= a_{00} + f^I + f^{II} + f^{III} \\ f^I &= a_{10}x + a_{01}y \\ f^{II} &= a_{20}x^2 + a_{11}xy - (1+a_{20})y^2 \\ f^{III} &= a_{30}(x^3 - 3xy^2) + a_{03}(y^3 - 3x^2y) \end{aligned}\right\} \quad (41)$$

If $y = mx + n$ is the equation of its asymptotes, we get m as a root of the cubic equation

$$a_{03}m^3 - 3a_{30}m^2 - 3a_{03}m + a_{30} = 0 \quad (42)$$

and n from the equation

$$3n(a_{03}m^2 - 2a_{30}m - a_{03}) = = (1+a_{20})m^2 - a_{11}m - a_{20} \quad (43)$$

When the 3 roots of Eq. (42) are real and different, we obtain three asymptotes. The curve then consists of 3 hyperbolic portions and of a closed oval (Group No. 1). Sometimes the oval is missing, as e.g. the curve

$$\left.\begin{aligned} y^2 &= (x^2-1)(1/3 - 1/x) \\ w/j &= x^3/3 - xy^2 - x^2 - x/3 + 1 \end{aligned}\right\} \quad (44)$$

This curve (Fig. 8) consists of 3 distinct portions, two of a hyperbolic nature and a third with a single asymptote. The flow is possible only within the right curve or between the other two curves. The isovel $w = j$ consists of the y-axis ($x = 0$) and the hyperbola

$$x^2 - 3y^2 - 3x - 1 = (x - 3/2)^2 - 3y^2 - 13/4 = 0 \quad (45)$$

which is of the type of Eq. (23).

POLYNOMIALS OF ORDER 4

We shall consider a few chosen examples.

Flow between two inclined planes of equations

$$y = 0, \quad mx - y = 0$$

As the velocity vanishes at those plane boundaries, the velocity can be written as

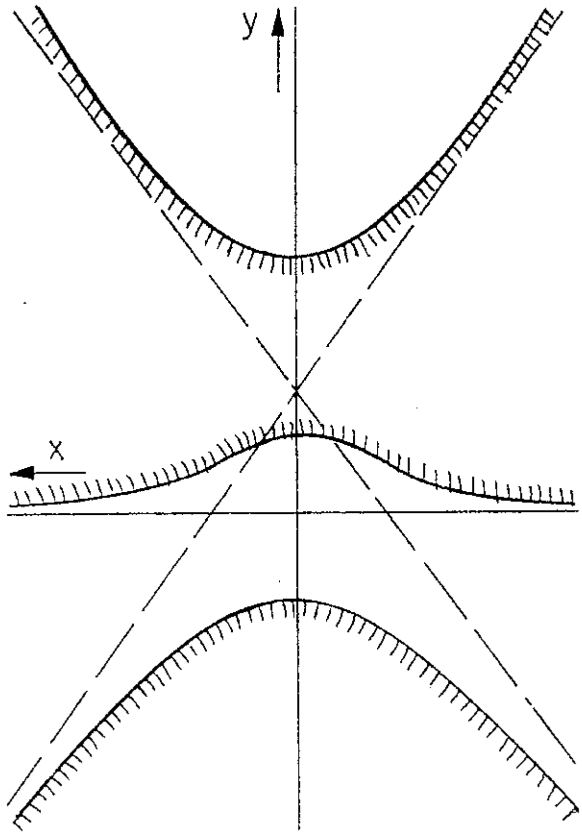

Fig. 8. Cubic.

$$\left.\begin{aligned} w/j &= f(x,y) = y(mx-y)(1+F^I+F^{II}+F^{III}+\cdots) \\ F^I &= a_1x + a_2y \\ F^{II} &= b_1x^2 + b_2xy + b_3y^2 \\ F^{III} &= c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 \end{aligned}\right\} \quad (46)$$

Let us introduce this into Eq. (5) and equate to 0 all coefficients.

For $\underline{f(x,y) = y(mx-y)(1+F^I)}$ we get (47)

$m = a_1/a_2 = \pm\sqrt{3}$, flow in an equilateral triangular prism.

For $\underline{f(x,y) = y(mx-y)(1+F^{II})}$ we get

$$m = \pm 1;\ f(x,y) = y(x \mp y)[b_2(xy \pm x^2) \pm 1] \quad (48)$$

Figure 9 shows the flow domain for $m = -1$, $b_2 = 1$. It describes flow between two planes at 45°, but which is completely different from the similar flow (28) for $m = -1$. It describes also flow inside the hyperbola with asymptotes enclosing an angle of 45°, which is completely different from a similar flow (25) inside the same hyperbola.

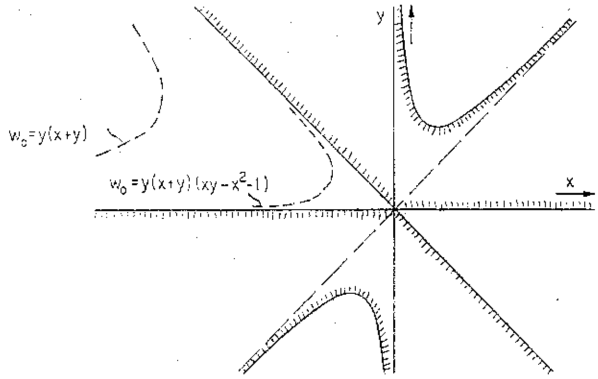

Fig. 9. Quartic.

As $b_2(xy \pm x^2) \pm 1$ is irreducible, there exists no algebraic solution of thé type

$$f = F_1{}^I F_2{}^I F_3{}^I F_4{}^I$$

For $f(x, y) = y(mx - y)(1 + f^{III})$ we get

$$\left.\begin{aligned} m &= \pm\sqrt{5 \pm 2\sqrt{5}} = \pm 3.078,\ \pm 0.726 \\ f(x, y) &= y(mx - y)\{1 + c_2[mx^3 + x^3 y \\ &\quad + (1/m - 2m)xy^2 + (2 - 1/m^2)y^3]\} \end{aligned}\right\} \quad (49)$$

This describes again flow between two planes at an angle $\tan^{-1} m$, which is completely different from a similar flow given by Eq. (28). Because of the different values of m for F^I, F^{II}, F^{III}, it is impossible to have a mixed solution of type

$$f(x, y) = y(mx - y)(1 + F^I + F^{II} + F^{III})$$

It can be shown that no algebraic flow of type F^{IV} is possible inside or outside an elliptic or parabolic cylinder. Therefore we shall consider flow inside or outside a hyperbolic cylinder.

FLOW WITH HYPERBOLIC BOUNDARIES

If the hyperbola has the equation

$$a + kx^2 - y^2 = 0 \quad (50)$$

the velocity can be put in the form

$$w/j = f(x, y) = (a + kx^2 - y^2)(1 + F^I + F^{II} + \cdots) \quad (51)$$

where $F^I, F^{II}, \cdots$ are given by (46). Introducing this into Eq. (5) and equating to 0 all coefficients of $x^m y^n$ we get successively the following solutions near the hyperbola:

$$w/j = a + kx^2 - y^2 \quad (52)$$

$$w/j = (a - 1.5x^2 + 0.5y^2)(1 + a_2 y) \quad (53)$$

$$\left.\begin{aligned} w/j &= [a + k(x^2 - \beta y^2)] \cdot [1 + b_1(x^2 - \beta' y^2)] \\ \beta &= 3 \mp \sqrt{8} = 0.172 \text{ (or } 5.828) \\ \beta' &= 3 \pm \sqrt{8} = 5.828 \text{ (or } 0.172) \\ k(3 &\mp 2\sqrt{2}) + ab_1 = 1 \mp \sqrt{2};\ \beta'\beta = 1 \end{aligned}\right\} \quad (54)$$

This procedure may be extended to higher orders, thus giving rise to new solutions.

UNSTEADY FLOW

The method outlined above can be extended to unsteady unidirectional flow inside or outside cylindrical ducts or surfaces. The partial differential equation, (4)

$$w_t = gJ(t) + \nu(w_{xx} + w_{yy}) \quad (55)$$

is of the parabolic type and J is a function of time only. The general algebraic solution of order N is

$$w = \sum_{m,n=0}^{m+n=k \leq N} a_{mn}(t) x^m y^n \quad (56)$$

Proceeding as above, with $\phi = 1/\nu, j = gJ\phi/2$

$$\left.\begin{aligned} a_{20} &= a'_{00}\phi/2 - j - a_{02} \\ a_{30} &= a'_{10}\phi/6 - a_{12}/3;\quad a_{03} = a'_{01}\phi/6 - a_{21}/3 \\ a_{40} &= a''_{00}\phi^2/24 + a'_{02}\phi/12 - a_{22}/6 \\ a_{13} &= a'_{11}\phi/6 - a_{31};\quad a_{04} = a'_{02}\phi/12 - a_{22}/6 \\ a_{50} &= a''_{10}\phi^2/120 + a'_{12}/60 - a_{32}/10 \\ a_{41} &= a'_{21}\phi/12 - a_{23}/2;\ a_{14} = a'_{12}\phi/12 - a_{32}/2 \\ a_{05} &= a''_{01}\phi^2/120 + a'_{21}\phi/60 - a_{23}/10 \end{aligned}\right\} \quad (57)$$

Here the coefficients a_{00}; a_{10}, a_{01}; a_{11}, a_{02}; a_{21}, a_{12}; a_{31}, a_{22}; $a_{32}, a_{23} \cdots$ are arbitrary. Equation (56) written explicitly to terms up to order 5 becomes

$$\begin{aligned} w &= a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) \\ &+ (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3) + \\ &+ (a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4) + \\ &+ (a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{14}xy^4 + \\ &+ a_{05}y^5) + \cdots \end{aligned} \quad (58)$$

The special case

$$w = a_{00} + F^I = a_{01}y$$

represents flow along the plane $y = 0$.

The special case

$$w = a_{00} + F^{II} = a_{00} + a_{20}x^2 + a_{02}y^2$$

represents flow inside an elliptic cylinder of semi-axes a, b, with

$$a_{00}(t) > 0; \quad a_{02}(t) = -a_{00}/b^2$$
$$a_{20}(t) = a'_{00}/2\nu - j - a_{02} = -a_{00}/a^2$$

CONCLUSIONS

1. New, relatively simple solutions of the Navier-Stokes equations can be found.

2. In the case of open domains unspecified at infinity different velocity distributions may exist for identical boundaries and longitudinal piezometric (or pressure) gradients. This fact should be considered when developing formulas for boundary layers.

3. In the case of open domains with narrow sections, the isovel pattern may show singularities, e.g. saddle points.

4. The above results may be applied to all cases obeying a similar linear equation of the Poisson type (Eq. 5): torsion of shafts of constant cross-section; deflection of thin membranes and soap bubbles; distribution of the shear stress in canals of constant cross-section. In particular the irreducible (or ineffective) porosity of a porous medium can be explained: this is the partial volume of the stagnant layer near the boundaries. When computed for some of the profiles studied here it gives a value of some 20%.

5. The method can be extended to unsteady flows. The number of possibilities is then considerably increased.

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