Fractals, Vol. 5, No. 4 (1997) 561–564

Letter

A LOOK AT FRACTAL FUNCTIONS THROUGH THEIR FRACTIONAL DERIVATIVES

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Received January 17, 1997; Accepted June 3, 1997

Abstract

A relationship between the Lipschitz-Holder exponent and the maximal possible order of derivative is used to look at fractal properties of functions representable by trigonometric series.

1. SOME PROPERTIES OF FRACTAL FUNCTIONS

A continuous function $y(x)$ is fractal if it is non-differentiable almost everywhere. The singular character of a fractal function is quantified by either one constant — its fractal dimension D (a number between 1 and 2), or in more detail by its multi-fractal singularity distribution function $f(\alpha)$, where f$\in$[0,1] is the fractal dimension of the set of points on x for which the Lipschitz-Holder (L-H) exponent of $y(x)$ is α. The most important point on the convex $f(\alpha)$ curve is its maximum α_o, where $f(\alpha_o) = 1$; $\alpha_o \in (0,1)$ is the most probable, L-H exponent.

Another important point on $f(\alpha)$ is the one related to the strongest singularity (smallest α), often $f(\alpha_{\min}) = 0$, where $0 < \alpha_{\min} \leq \alpha_o$. Always $\alpha_{\min} \leq$ (2-D) $\leq \alpha_o$, and quite often (2-D) is rather close to α_o.

2. TRIGONOMETRIC SERIES

In the sequel we confine the treatment to functions expressible in a form of an infinite trigonometric series, such as:

$$y(x) = \sum_{n=1}^{\infty} a_n \cos(b_n x + \varphi_n) \qquad (1)$$

where it is assumed that: (i) the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent; (ii) the sequence of positive b_n in (1) increases monotonically; and (iii) φ_n are arbitrary phase-shifts, given or random.

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Oldham & Spanier (1974) is given by:

$$\underset{x_o x}{D^s} y = \lim_{m\to\infty}\left(\frac{x-x_0}{m}\right)^{-s}\sum_{k=0}^{m-1}(-)^k\binom{s}{k}x$$

$$y\left(x-\frac{k(x-x_o)}{m}\right) \tag{4}$$

For $s = 0,1,2,\dots \underset{x_o x}{D^s y}$ boils down to $y(x), y'(x)$ $y''(x),\dots$. In the sequel we shall make use of the fractional derivation of powers and cosines

$$\underset{x_o x}{D^s}(x-x_o)^\omega = \frac{\Gamma(\omega+1)}{\Gamma(\omega-s+1)}(x-x_o)^{\omega-s} \tag{4a}$$

$$\underset{-\infty x}{D^s}\, a\cdot\cos(kx+\varphi) = a\cdot k^s\cos\left(kx+\frac{s\pi}{2}+\varphi\right) \tag{4b}$$

Although generally $s\in R$ we will concentrate on $s\in[0,1]$. The s derivatives of the series Eq. (1) is given by:

$$\underset{-\infty x}{D^s} y = \sum_{n=1}^{\infty} a_n b_n^s \cos\left(b_n x+\frac{s\pi}{2}+\varphi_n\right) \tag{5}$$

It is clear that the existence of $\underset{-\infty x}{D^s} y$ depends on the convergence of $\sum_{n=1}^{\infty} a_n b_n^s$, which depends on s.

4. THE α/s CONJECTURE

We conjecture that the maximum possible order of derivative s_{sup} of any continuous $y(x)$ at a point x, (i.e. $\underset{x_o x}{D^s} y$, is finite for all $0\le s < s_{\text{sup}}$), is equal to the L-H exponent α at the same point: $\alpha = s_{\text{sup}}$.

Explanation:

A typical behaviour of $y(x)$ at the vicinity of the point x_1 with L-H exponent $\alpha(x_1)=\alpha_1$ is:

$$y-y_1 = y(x)-y(x_1)\approx (x-x_1)^{\alpha_1}\cos(\ell n(x-x_1))$$

$$= \text{Re}\,\{(x-x_1)^{\alpha_1+i}\} \tag{6}$$

(see West (1990), p. 78.)
The s derivative of Eq. (6), as given by Eq. (4a), is

$$\underset{x_o x}{D^s}(y-y_1) = \text{Re}\left\{\frac{\Gamma(\alpha_1+i+1)}{\Gamma(\alpha_1+i-s+1)}(x-x_1)^{\alpha_1-s+i}\right\}$$

$$= \text{Re}\left\{\frac{\Gamma(\alpha_1+i+1)}{\Gamma(\alpha_1+i-s+1)}(x-x_1)^{\alpha_1-s} \times e^{i\ell n(x-x_1)}\right\} \tag{7}$$

The maximum value of s for which the above converges is $s_{\text{sup}}=\alpha_1$.

5. THE TRIGONOMETRIC SERIES CONJECTURE

Given that $s_{\min}, s_o$ and $s_{\max}$ are the largest values of s for which the following series of numbers are bounded, respectively:

$$S_{(+)} = \sum_{n=1}^{\infty} a_n b_n^s \tag{8a}$$

$$S_o = \sum_{n=1}^{\infty}(a_n b_n^s)^2 \tag{8b}$$

$$S_{(-)} = \sum_{n=1}^{\infty}(-)^n a_n b_n^s \tag{8c}$$

We conjecture that the most probable L-H exponents α_o and smallest L-H exponent $\alpha_{\min}$ (strongest singularity) are bounded by:

$$\text{(i) } s_{\min}\le\alpha_{\min}\le s_o \tag{9a}$$

$$\text{(ii) } s_o\le\alpha_o\le s_{\max} \tag{9b}$$

Explanation:

Based on the relationship between the local α and local s_{sup} as stated in the α/s conjecture, and on the derivative $\underset{-\infty x}{D^s} y$ given by Eq. (5), it is plausible that $s_{\min}$ and $s_{\max}$ should set the extreme bounds on $\alpha_{\min},\alpha_o$; i.e. that $s_{\min}\le\alpha_{\min}\le\alpha_o\le s_{\max}$. Note that taking all cosines in Eq. (5) to be equal one (as in Eq. (8a)), definitely yields the worst conditions for its convergence; whereas the alternating signs in Eq. (8c) are probably the best choice for its convergence.

The importance of s_o is obtained by treating the arguments of the cosines in Eq. (5) as random variables. The central limit theorem then tells us that the average of $\underset{-\infty x}{D^s} y$ is zero and that its variance is finite only if S_o is finite. If s_o is the largest value of s for which Eq. (8b) exists, then a random function y (with random phases) should have an s_o derivative almost everywhere.

The derivative of $y(x)$, as obtained by term by term differentiation of Eq. (1), exists everywhere only when the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent. In the present note we focus on non-differentiable (fractal) functions for which $\sum_{n=1}^{\infty} a_n b_n$ diverges.

Two "well-known" members of this class of functions are Weierstrass' and Riemann's non-differentiable functions given, respectively, by:

$$W(x; d, \gamma) = \sum_{n=1}^{\infty} \gamma^{(d-2)n} \cos(\gamma^n x), \quad \gamma > 1,\ 2 > d > 1 \qquad (2)$$

$$R(x; \mu) = \sum_{n=1}^{\infty} n^{-2\mu} \cos(n^2 x), \quad 1 > \mu > 1/2 \qquad (3)$$

Figures 1 and 2 show the Weierstrass function with $d = 1.35$ and $\gamma = 2$, and Riemann's function with $\mu = 0.75$, respectively. Both graphs have a fractal dimension $D \approx 1.35$, the first is a monofractal, and the second a multifractal, as explained later.

3. FRACTIONAL DERIVATIVES

Fractional calculus provides us with a generalization of the concept of differentiation. It enables to take a fractional (i.e. broken, non-integer) derivative of a function. The definition of the fractional derivative of arbitrary order s over (x_o, x), following

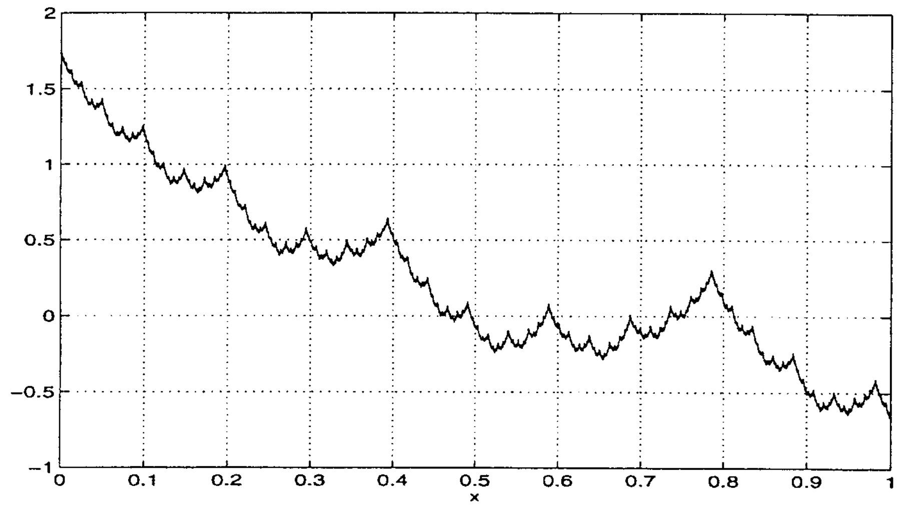

Fig. 1 Weierstrass' function with $d = 1.35$ and $\gamma = 2$.

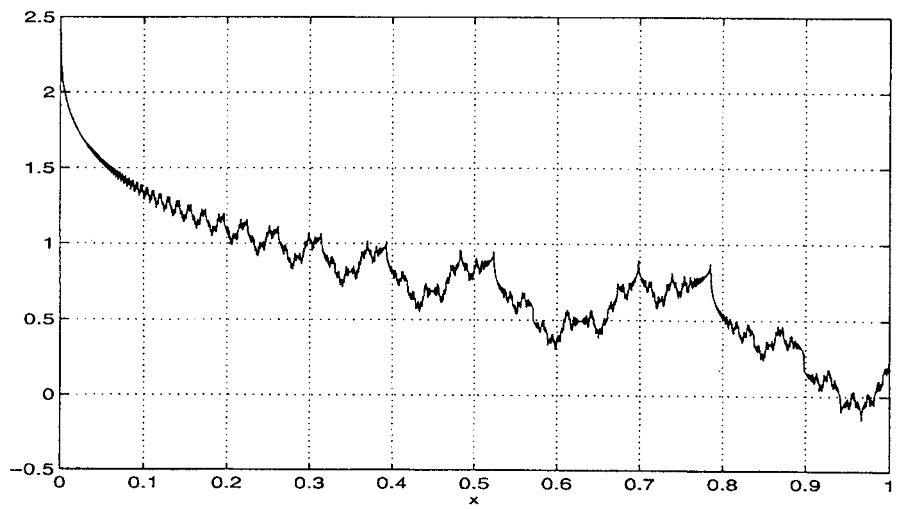

Fig. 2 Riemann's function with $\mu = 0.75$.

6. APPLICATIONS

Weierstrass function

From Eq. (2) and the trigonometric series conjecture we have

$$S_{(+)} = \sum_{n=1}^{\infty} \gamma^{n(d-2+s)}$$

$$S_0 = \sum_{n=1}^{\infty} \gamma^{2n(d-2+s)}$$

$$S_{(-)} = \sum_{n=1}^{\infty} (-)^n \gamma^{n(d-2+s)}$$

For convergence we need negative exponents, thus,

$$s_{\min} = s_o = s_{\max} = 2-d; \quad \text{and} \quad \alpha_o = \alpha_{\min} = 2-d\,.$$

This result is an indication that W is a mono-fractal and that its dimension D=d.

Riemann's function

From Eq. (3) and the trigonometric series conjectures, we have

$$S_o = \sum_{n=1}^{\infty} (n^{-2\mu} \cdot n^{2s})^2 = \sum_{n=1}^{\infty} n^{4(s-\mu)}$$

$$S_{(+)} = \sum_{n=1}^{\infty} (n^{-2\mu} \cdot n^{2s}) = \sum_{n=1}^{\infty} n^{2(s-\mu)}$$

$$S_{(-)} = \sum_{n=1}^{\infty} (-)^n n^{-2\mu} n^{2s} = \sum_{n=1}^{\infty} (-)^n n^{2(s-\mu)}$$

For convergence we need the exponents to be less than -1 for S_o and $S_{(+)}$; and less than zero for $S_{(-)}$. Thus, $s_o = \mu - 1/4$, $s_{\min} = \mu - 1/2$; and $s_{\max} = \mu$; so that $\mu > \alpha_o > \mu - 1/4$; and $\mu - 1/4 > \alpha_{\min} > \mu - 1/2$ in accordance with the numerical calculation for $0.5 < \mu < 1$, in Stiassnie & Hadad, 1995. The multifractal character of Riemann's function is evident.

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