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Shoaling of nonlinear wave-groups on water of slowly varying depth

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1. Introduction

The shoaling of weakly nonlinear surface wave groups is important to the understanding of coastal wave climate and coastal flow regime.

In the past, most efforts concentrated on the equally important though simpler problem of shoaling of wave-trains (i. e. *monochromatic* wave groups), for details see Stiassnie Peregrine (1980).

The first mathematical formulation for shoaling of wave-groups was given by Djordjevic' and Redekopp (1978), and in a somewhat improved version by Stiassnie (1983). This formulation is limited to cases where the water depth is small compared to the group-length. Equations suitable for water depths of the order of the group-length are deduced in Peregrine (1983); combining the constant depth model by Davey and Stewartson (1974) and the higher-order model for infinitely deep water by Dysthe (1979).

The only available solutions are those for the shoaling of isolated wave-packets (solitons), which were originally given by Djordjevic' and Redekopp in their 1978 paper. They predict that a soliton envelope can undergo fission only if it propagates into deeper water. By heuristic assumptions for the evolution along the slope, they also estimate the number of solitons emitted after a single soliton descends from a shallower shelf. A more recent study, Turpin, Benmoussa and Mei (1983) confirms these results qualitatively, but not quantitatively.

To the best of our knowledge, no results for shoaling of wavegroups (i. e. *modulated* wave-trains) have been presented so far. These modulated wave-trains are of particular importance since almost every wave-train will eventually become modulated due to its intrinsic Benjamin-Feir instability. The aim of the present paper is to throw some light on the evolution during the shoaling of a modulated wave-train and its influence on the mean free surface and the wave-induced mean flow.

Section 2, 3 and 4 outline the derivation of the mathematical model and its simplifications to a level which enables an analytical solution. The model presented at the end of Sect. 2 is rather general; it consists of a coupled system of equations for the complex wave envelope and the induced mean flow potential. In Sect. 3 we add the assumption of periodicity of the modulation, which leads to decoupling of the system of equations. The result is a nonlinear Schrödinger equation with a variable (depth dependent) coefficient, which can be solved by reasonable numerical efforts. In Sect. 4 we adopt the assumption of *three-waves sytems*, which has been used for constant depth in Stiassnie and Kroszynski (1982), and which together with a W.K.B. type approach enable an analytic, though asymptotic solution. This asymptotic solution is compared with numerical results in Sect. 5. Sections 6, 7 and 8 include a detailed presentation and discussion of the physical results obtained from our calculations.

2. Evolution equations

Assuming irrotational motion, there exists a velocity potential $\varphi(x, z, t)$ which satisfies Laplace's equation:

$$\varphi_{xx} + \varphi_{zz} = 0, \tag{2.1}$$

t is time, x is the horizontal coordinate in the direction of wave propatation, and z is the vertical coordinate pointing upward from the undisturbed free surface.

The boundary condition on the bottom, $z = -h(x)$, is

$$\varphi_z = -h'(x)\,\varphi_x. \tag{2.2}$$

The boundary conditions on the free surface, $z = \zeta(x, t)$, are the kinematic condition:

$$\varphi_z = \zeta_t + \varphi_x \zeta_x, \tag{2.3}$$

and the dynamic condition:

$$2g\zeta + 2\varphi_t + \varphi_x^2 + \varphi_z^2 = 0. \tag{2.4}$$

Considering situations for which the depth $h(x)$ as well as the wave input vary slowly, we assume all wave-field properties to be slowly changing. To render the term "slowly" explicit we introduce a small nondimensional parameter ε which is a measure of the wavy surface slope, and define the following new variables:

$$\tau = \varepsilon\left(\int^{x} \frac{\mathrm{d}x}{\Omega'} - t\right), \tag{2.5 a}$$

$$\xi = \varepsilon^2 x; \tag{2.5 b}$$

where $\Omega' = \partial\Omega/\partial K$ is the group velocity. $\Omega^2 = g\,K\,t\,h(K\,h)$ is the linear dispersion relation, relating the leading order constant frequency Ω to the leading order wave-number $K(\xi)$ and the water depth $h(\xi)$.

Having the wave-groups (i. e. wave fields with narrow spectra) in mind the velocity potential φ and the free surface displacement ζ are expanded in Fourier series:

$$\varphi = \varphi_0(x, z, t) + \{\varphi_1(\tau, \xi, z)\, e^{i\theta} + \varphi_2(\tau, \xi, z)\, e^{2i\theta} + \cdots \text{c.c.}\} \tag{2.6 a}$$

$$\zeta = \zeta_0(\tau, \xi) + \{\zeta_1(\tau, \xi)\, e^{i\theta} + \zeta_2(\tau, \xi)\, e^{2i\theta} + \cdots + \text{c.c.}\} \tag{2.6 b}$$

where $\theta = (\int^x K(\xi)\,dx - \Omega t)$, and c. c. stands for the complex conjugate. With ε chosen to be small, the functions $\varphi_j(\tau, \xi, z)$ and $\zeta_j(\tau, \xi)$ for $j \geqq 1$ are expanded formally in power series of ε:

$$\varphi_j(\tau, \xi, z) = \sum_{m=j}^{\infty} \varepsilon^m \varphi_{mj}(\tau, \xi, z) \tag{2.7 a}$$

$$\zeta_j(\tau, \xi) = \sum_{m=j}^{\infty} \varepsilon^m \zeta_{mj}(\tau, \xi). \tag{2.7 b}$$

The mean water level ζ_0, and the induced mean flow potential φ_0, require a special treatment and are best written as:

$$\zeta_0(\tau, \xi) = \varepsilon^2 \zeta_{20}. \tag{2.7 c}$$

$$\varphi_0(x, z, t) = \varepsilon\, \varphi_{10}(\tau, \xi, z) + \varepsilon^2 \varphi_{20}(\xi) \cdot t. \tag{2.7 d}$$

Note that it turns out that while φ_{mj} and ζ_{mj} *for* $j \geqq 1$ are all of order one; the order of φ_{10}, ζ_{20} and φ_{20} is between one for $Kh = 0(1)$ and ε for $Kh \to \infty$. The term $\varepsilon^2 \varphi_{20}(\xi) \cdot t$, in Eq. (2.7 d), is needed to suppress terms that grow boundlessly with time at higher order, as was shown in Stiassnie (1983). Following the method of derivation used by Djordjevic' and Redekopp (1978) but using Eqs. (2.7 c, d), instead of expanding φ_0 and ζ_0 in power series of ε (which is justified only for $Kh = 0(1)$), we obtain a system of evolution Eqs. (2.9 a, b, c, d) for φ_{10} and the complex wave envelope,

$$A(\tau, \xi) = -2i\varepsilon\zeta_{11} \tag{2.8}$$

as follows:

$$\frac{i}{2\Omega'}\frac{\partial\Omega'}{\partial\xi} A + iA_\xi + \frac{\Omega''}{2(\Omega')^3} A_{\tau\tau} - \frac{\varepsilon^{-2}\beta_1}{\Omega'} |A|^2 A = (\beta_2 \varphi_{10_\tau} + \beta_3 \varphi_{20}) \frac{A}{\Omega'}; \quad z = 0, \tag{2.9 a}$$

$$\varphi_{10_{zz}} + \frac{\varepsilon^2}{(\Omega')^2} \varphi_{10_{\tau\tau}} = 0; \quad -h \leqq z \leqq 0, \tag{2.9 b}$$

$$\varphi_{10_z} + \frac{\varepsilon^2}{g} \varphi_{10_{\tau\tau}} = \frac{g\beta_2}{2\Omega} (|A|^2)_\tau; \quad z = 0 \tag{2.9 c}$$

$$\varphi_{10_z} = 0; \quad z = -h. \tag{2.9 d}$$

The depth dependent coefficients are:

$$\beta_1 = \frac{g K^3}{2\Omega} \cdot \frac{9 - 12\, th^2(Kh) + 13\, th^4(Kh) - 2\, th^6(Kh)}{8\, th^3(Kh)}, \tag{2.10a}$$

$$\beta_2 = \frac{K^2}{2\Omega} \cdot \left(\frac{2\Omega}{K\Omega'} + sech^2(Kh)\right), \tag{2.10b}$$

$$\beta_3 = \frac{K^2}{2\Omega} sech^2(Kh). \tag{2.10c}$$

A set of modulation equations equivalent to (2.9), but for constant depth, has been recently derived from the finite depth Zakharov equation, see Stiassnie and Shemer (1984). The mean water level ζ_0 is given by:

$$\zeta_0 = \frac{\varepsilon^2}{g} (\varphi_{10_\tau} - \varphi_{20}) - \frac{g K^2}{4\Omega^2} sech^2(Kh) \cdot |A|^2 \tag{2.11}$$

and is of order between ε^2 to ε^3, depending on the water depth.

3. Periodic cases

Restricting the discussion to cases for which the complex wave envelope $A(\tau, \xi)$ is periodic in τ, and assuming zero averaged (over τ) mass flow in the x-direction (as in the case of an impermeable beach) enables the decoupling of the system (2.9). For these cases A is governed by the nonlinear Schrödinger equation:

$$\frac{i}{2\Omega'} \frac{\partial \Omega'}{\partial \xi} A + i A_\xi + \frac{\Omega''}{2(\Omega')^3} A_{\tau\tau} - \frac{\varepsilon^{-2}\alpha_1}{\Omega'} |A|^2 A = \frac{\varepsilon^{-2}\alpha_2}{\Omega'} \overline{|A|^2} A, \tag{3.1}$$

where the bar denoted averaging over τ, and

$$\alpha_1 = \beta_1 + \frac{g^2 \beta_2^2}{2\Omega(1 - gh/(\Omega')^2)}, \tag{3.2a}$$

$$\alpha_2 = -\frac{g K \Omega'}{2h\Omega}(\beta_2 + \beta_3) - \frac{g^2 \beta_2^2}{2\Omega(1 - gh/(\Omega')^2)}. \tag{3.2b}$$

The induced mean flow potential is given by:

$$\varphi_{10} = \frac{-\varepsilon^{-2}\overline{|A|^2} g K \Omega'}{2\Omega h} \cdot \tau + \sum_{n=1}^{\infty} a_n(z) \{b_n(\xi) e^{2\pi i n \tau/\gamma} + b_{-n}(\xi) e^{-2\pi i n \tau/\gamma}\} \tag{3.3}$$

where γ is the period of $A(\tau, \xi)$;

$$a_n = \cosh(2\pi\varepsilon n(z+h)/\Omega'\gamma), \quad n = 1, 2, \ldots, \tag{3.3 a}$$

$$b_n = \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega\left\{\frac{2\pi n g}{\varepsilon\Omega'\gamma}\sinh\left(\frac{2\pi n\varepsilon h}{\Omega'\gamma}\right) - \left(\frac{2\pi n}{\gamma}\right)^2 \cosh\left(\frac{2\pi n\varepsilon h}{\Omega'\gamma}\right)\right\}} \cdot C_n, \tag{3.3 b}$$

and C_n are the Fourier coefficients of $(|A|^2)_\tau$:

$$(|A|^2)_\tau = \sum_{n=1}^{\infty} \{C_n(\xi)\, e^{2\pi i n\tau/\gamma} + C_{-n}(\xi)\, e^{-2\pi i n\tau/\gamma}\}. \tag{3.4}$$

The convergence of the Fourier series (3.4) was assumed to be independent of ε.

The potential φ_{20} which is needed to calculate the mean water level ζ_0, see Eq. (2.11), is given by:

$$\varphi_{20} = -\varepsilon^{-2}\,\overline{|A|^2}\, g K \Omega'/2\Omega h. \tag{3.5}$$

The details of the derivation of (3.1) to (3.5) are given in Appendix *A*. Eq. (3.1) is identical to Eq. (4.9) in Stiassnie (1983), which has been derived assuming no vertical dependence of the induced mean flow. The latter assumption is strictly valid only when $Kh = 0(1)$, namely, for cases where the water depth h is small compared to the group length $(\varepsilon K)^{-1}$. For these cases the present $\varphi_{0,x}$ and ζ_0 yield Eq. (3.7) and Eq. (3.8) of Stiassnie (1983).

A simpler dimensionless form of Eq. (3.1) is obtained by means of the transformations

$$\psi = \varepsilon^{-1}\left(\frac{2\Omega^5\Omega'}{g^3}\right)^{1/2} A \exp i\left(\overline{|A|^2}\int_{x_\infty}^{x}\frac{\alpha_2\, dx}{\Omega'}\right), \tag{3.6 a}$$

$$T = \tau/\gamma, \quad X = \frac{1}{\gamma^2}\int_{\xi_\infty}^{\xi}\frac{\Omega''}{2(\Omega')^3}\, d\xi. \tag{3.6 b}$$

which give

$$i\psi_X + \psi_{TT} + \mu|\psi|^2\psi = 0, \tag{3.7}$$

$$\mu(\xi) = \frac{-g^3\Omega'\gamma^2\alpha_1}{\Omega^5\Omega''}. \tag{3.8}$$

The dimensionless parameter μ is a monotonic increasing function of Kh, having the values zero and $(\Omega\gamma)^2$ for $Kh = 1.363$ and $Kh \to \infty$ respectively. The statement of the mathematical problem, given by Eq. (3.7), is completed by the following input condition at $X = 0$ (i.e. $x = x_\infty$: a reference point in infinitely deep water).

$$\psi(T, 0) = 1 + 2\beta\, e^{i\alpha}\cos(2\pi T) \tag{3.9}$$

which corresponds to a system composed of a carrier-wave and a symmetric "side-band" disturbance

$$\zeta(t, x_\infty) = \frac{g\,\varepsilon}{\Omega^2} \operatorname{Re}\{e^{-i\Omega t} + \beta\, e^{-i[(1+2\pi\varepsilon/\Omega\gamma)\Omega t - \alpha]} + \beta\, e^{-i[(1-2\pi\varepsilon/\Omega\gamma)\Omega t - \alpha]}\} + 0(\varepsilon^2). \tag{3.10}$$

For constant depth, $\mu = \text{const}$, it is well-known that Eq. (3.7) with T in (0, 1), subject to periodic boundary conditions has the following X invariants

$$J_1 = \int_0^1 |\psi|^2 \, \mathrm{d}T, \tag{3.11 a}$$

$$J_2 = \int_0^1 (\psi^* \psi_T - \psi \psi_T^*) \, \mathrm{d}T, \tag{3.11 b}$$

$$J_3 = \int_0^1 \left(|\psi|^4 - \frac{2}{\mu} |\psi_T|^2\right) \mathrm{d}T. \tag{3.11 c}$$

These invariants are determined by the input condition, Eq. (3.9), so that

$$J_1 = 1 + 2\beta^2, \qquad J_2 = 0, \tag{3.12 a, b}$$

$$J_3 = 1 + (4 - P + 2\cos 2\alpha) \cdot 2\beta^2 + 6\beta^4 \tag{3.12 c}$$

where

$$P = 8\pi^2/\mu. \tag{3.13}$$

For varying depth, $\mu = \mu(X)$, J_1 and J_2 remain invariant and are given by (3.11 a, b) and (3.12 a, b), but J_3 is a function of X governed by the equation,

$$\frac{\mathrm{d}J_3}{\mathrm{d}X} = \frac{2\mu_X}{\mu^2} \int_0^1 |\psi_T|^2 \, \mathrm{d}T. \tag{3.14}$$

4. Three-waves systems

The solution of Eq. (3.7) can be expanded in a Fourier series

$$\psi(T, X) = \sum_{n=-\infty}^{\infty} D_n(X)\, e^{2\pi i n T}. \tag{4.1}$$

The boundary condition at $X = 0$, Eq. (3.9), gives $D_0(0) = 1$; $D_1(0) = D_{-1}(0) = \beta\, e^{i\alpha}$; $D_n(0) = D_{-n}(0) = 0$ for $n \geqq 2$.

Stiassnie and Kroszynski (1982) truncated the above given series and considered only *three waves systems*:

$$\psi(T, X) = \sum_{n=-1}^{1} D_n(X)\, e^{2\pi i n T}. \tag{4.2}$$

Substituting Eq. (4.2) into Eq. (3.7) yields the following system of *ordinary differential* equations:

$$i\frac{\mathrm{d}D_0}{\mathrm{d}X} + \mu\,[(|D_0|^2 + 4\,|D_1|^2)\,D_0 + 2\,D_1^2\,D_0^*] = 0, \tag{4.3 a}$$

$$i\frac{\mathrm{d}D_1}{\mathrm{d}X} + \mu\left[\left(2\,|D_0|^2 + 3\,|D_1|^2 - \frac{P}{2}\right)D_1 + D_0^2\,D_1^*\right] = 0. \tag{4.3 b}$$

Note that Eq. (3.11 b) yields $D_{-1} = D_1$.

For constant depth the system of Eqs. (4.3 a, b) has exact solutions in terms of Jacobian elliptic functions with periods of order 1 in X which are summarized in Appendix B; for details see Stiassnie and Kroszynski (1982). These solutions depend on the invariants J_1, J_3, and on the parameter μ, which in turn depends on the water depth h and on the modulation period γ. For very mild depth variations, where $h_X = o(1)$, we apply an asymptotic, *WKB* related approach, assuming the local solution to be that of the constant depth type and using Eq. (3.14) to determine J_3. J_1 and γ are fixed by the input conditions and J_3, is given through $I(P)$ by:

$$\frac{\mathrm{d}I}{\mathrm{d}P} = -\sqrt{\frac{P(4-P)}{7}}\;\frac{\left(2\ln\dfrac{1+\sqrt{\dfrac{4-P}{7P}}}{1-\sqrt{\dfrac{4-P}{7P}}}\right)}{\ln\left(\dfrac{2\,P^2\,(4-P)^2}{|(2\,P-1)\,I|}\right)} \tag{4.4}$$

where

$$I(P) = J_3(P) - J_1^2. \tag{4.5}$$

The initial value of I, at $X = 0$, where $P = P_0$ is denoted by I_0 and is given by:

$$I_0 = \beta^2\,[2\,\beta^2 + 4\,(1 + \cos 2\,\alpha) - 2\,P_0]. \tag{4.6}$$

I_0, as well as $I(P)$ were assumed to be $o(1)$ throughout the rather lengthy derivation of Eq. (4.4). In all our examples we choose ($\gamma = 2\,\pi\,\Omega^{-1}$) $P_0 = 2$, corresponding to the fastest growth-rate of the Benjamin-Feir instability.

5. Numerical verification of the asymptotic solution

In order to appraise the relevance of the asymptotic solution given in the previous section, we compare its results with those of a numerical solution of the system of ordinary differential Eq. (4.3 a, b).

Figure 5.1 shows $I = I(P)$ for four initial values of $I_0 = I(2)$ ($I_0 = -0.04$, -0.01, 0.04, and 0.1). The broken line represents the asymptotic solution and was obtained by numerical integration of Eq. (4.4). The solid line was obtained, by substitution of the results obtained from a numerical solution of the system of O.D.E (4.3 a. b) into the expression

$$I(P) = 2|D_1|^2 \{|D_1|^2 + 2|D_0|^2 - P + 2|D_0|^2 \cos[2(\arg D_1 - \arg D_0)]\}. \tag{5.1}$$

The numerical solution of the system of O.D.E was obtained using a trapezoidal method and assuming the $P(h(X)) = 2 + 0.2X$. Note that the assumption $|I| \ll 1$, which is necessary for the asymptotic solution to be valid, imposes a restriction on the range of variation of P ($P = 2.8$ corresponds to $\Omega^2 h/g = 4$). In Fig. 5.2 we show three parts of the *exterior group envelope* $|\psi(0, X)|$ as well as the *interior group envelope* $|\psi(\frac{1}{2}, X)|$ for the input conditions $\alpha = 0$, $\beta = 0.158$ ($I_0 = 0.1$).

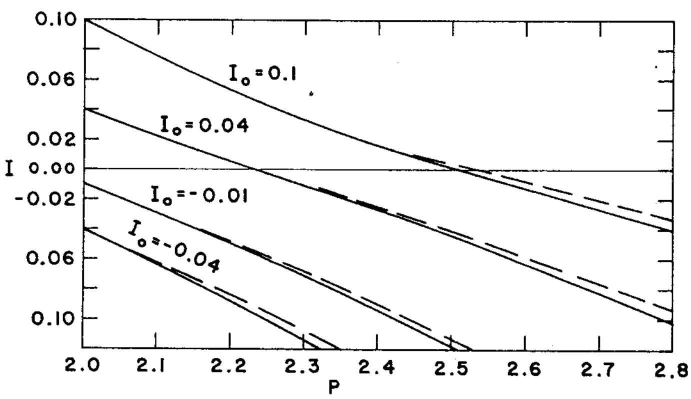

Figure 5.1
$I = I(P)$, — — — Asymptotic solution, ——— Numerical solution of Eqs. (4.3 a, b).

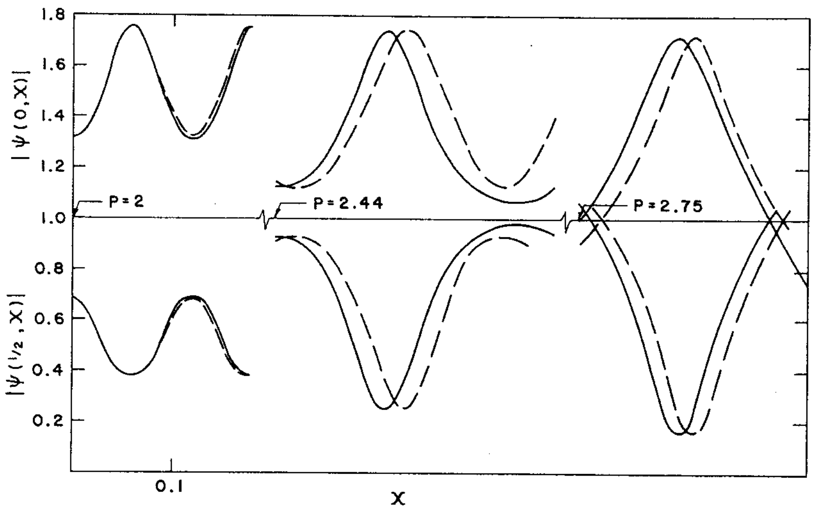

Figure 5.2
Exterior-group-envelope $|\psi(0, X)|$ and Interior-group-envelope $|\psi(\frac{1}{2}, X)|$ for the input conditions $\alpha = 0$, $\beta = 0.158$, ($I_0 = 0.1$) — — — — Asymptotic solution, ——— Numerical solution of Eqs. (4.3 a, b).

Here again, solid lines represent the numerical solution of Eqs. (4.3 a, b) with $P = 2 + 0.2\,X$ while the broken lines correspond to results obtained by the asymptotic method, utilizing the relation:

$$|\psi(T, X)|^2 = \{-2I + 2(4J_1 - P)\tilde{z} - 7\tilde{z}^2 + [S(2I + 2P\tilde{z} - \tilde{z}^2)^{1/2} + 4\tilde{z}\cos 2\pi T]^2\}/8\tilde{z} \qquad (5.2)$$

where $\tilde{z}$ is given in Appendix B, and S is the sign of $\cos\alpha$.

The three parts shown in Fig. 5.2 are for $P = 2, 2.44, 2.75$ for the asymptotic solution compared to P in (2, 2.03), (2.44, 2.49), (2.75, 2.81) for the numerical solution of the O.D.E., respectively. The agreement between the two methods of solution, as seen in both the above figures is rather encouraging and seems to indicate the validity of our new asymptotic solution of the system (4.3 a, b). Nevertheless, one still has to answer the question if, and to what extent, the system (4.3 a, b) itself is a reasonable substitute for the N.L.S, Eq. (3.7). For constant depth Stiassnie and Kroszynski (1982) show a good quantitative agreement in the length of the modulation-demodulation cycle and only a qualitative agreement for the amplitudes. A similar trend can be seen in Fig. 5.3, for $X < 0.6$, which compares two numerical solutions, one for the N.L.S. (3.7) – dotted line, and the other for the system of O.D.E. (4.3 a, b) – solid line.

For $X > 0.66$ the exterior and interior group envelopes, given by the N.L.S. (3.7), start crossing each other and interchanging their roles. This trend is reproduced by the solution of the system of O.D.E. (4.3 a, b) three modulation-demodulation cycles downstream, starting at $X \geqq 1.23$.

The input data in Fig. 5.3 is $\alpha = 0$, $\beta = 0.1$ ($I_0 = 0.04$) and the variation $P = 2 + 0.2\,X$ is assumed. Both the exterior and interior group envelopes are drawn.

We believe that our *much-simplified* asymptotic solution is not over-simplified, and is able to produce quite a few results of qualitative, and maybe even semi-quantitative relevance, which enable us some new physical insight.

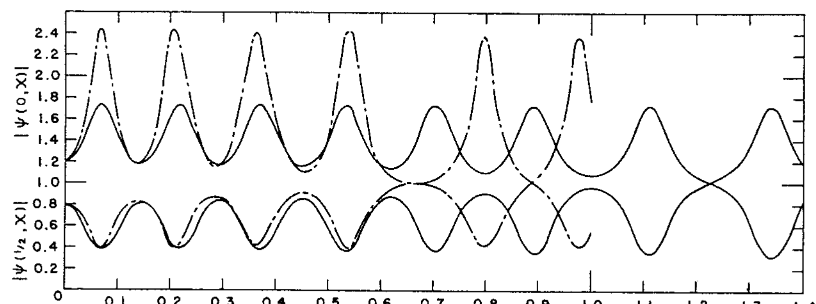

Figure 5.3
Exterior-group-envelope $|\psi(0, X)|$ and Interior-group-envelope $|\psi(\frac{1}{2}, X)|$ for the input conditions $\alpha = 0$, $\beta = 0.1$, ($I_0 = 0.04$), ——— Numerical solution of O.D.E (4.3 a, b) — · — · Numerical solution of N.L.S. (3.7).

6. On P and I_0

One fundamental property of the asymptotic solution is that it depends on P and I_0 solely. Given the input data α, β (and $P_0 = 2$), I_0 is determined by Eq. (4.6). Then, integrating Eq. (4.4) from P_0 to P the parameter $I(P)$ is found, and the solution given by Stiassnie and Kroszynski (1982), is locally applied.

Figure 6.1 gives the relation between P and nondimensional local water depth $K_\infty h$, (where $K_\infty = \Omega^2/g$ is the wave-number in infinitely deep water), for $\gamma = 2\pi\Omega^{-1}$. P is a monotonic decreasing function having the values infinity at $K_\infty h = 1.195$ ($Kh = 1.363$) and 2 for $K_\infty h \to \infty$. The input value I_0, (for $P_0 = 2$) dependence on α and β is shown in Fig. 6.2. Note that different combinations of α and β give the same I_0, and thus basicly the same solution for any P.

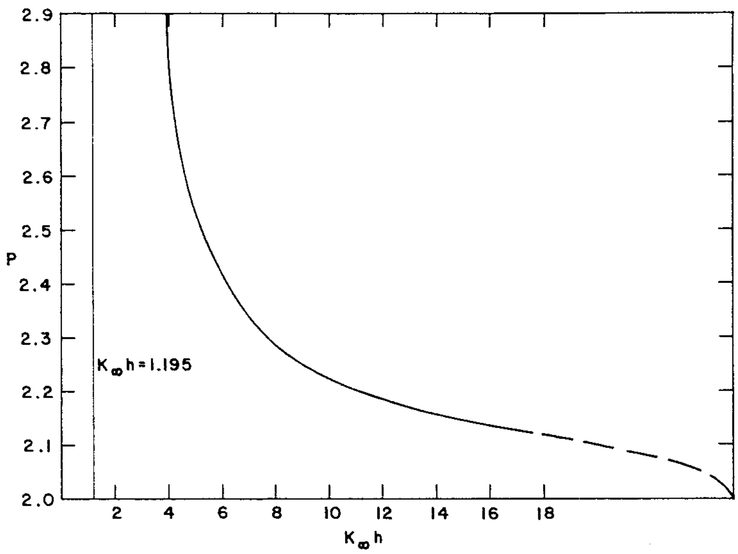

Figure 6.1
$P = P(K_\infty h)$ for $P_0 = 2$.

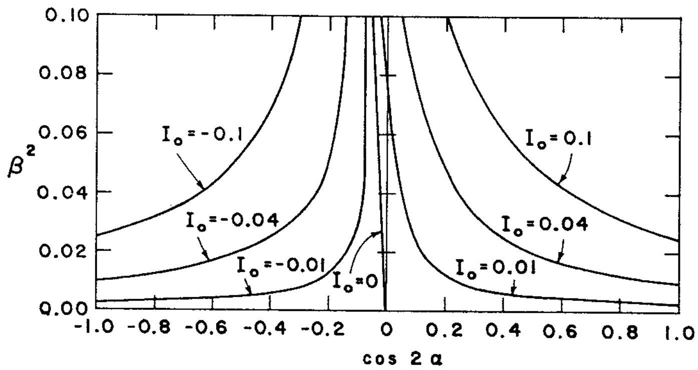

Figure 6.2
$I_0 = I_0(\alpha, \beta)$ for $P_0 = 2$
(on $\beta^2 = 0$: $I_0 = 0$).

7. Group envelopes

The free-surface of an (unstable) shoaling wave-train, displays three distinct length scales: λ_1 – the *wave length*; λ_2 – the modulation or *group length*; and λ_3 – the modulation-demodulation, group-envelope, or maybe best "*supergroup*" *length*. These three lengths are given by

$$\lambda_1 = 2\pi/K, \tag{7.1 a}$$

$$\lambda_2 = 2\pi\varepsilon^{-1}\Omega'/\Omega, \tag{7.1 b}$$

$$\lambda_3 = \frac{4\varepsilon^{-2}P(\Omega')^3}{\Omega^2\Omega''\sqrt{P(4-P)}}\ln\left|\frac{2P^2(4-P)^2}{(2P-1)I}\right|, \tag{7.1 c}$$

see Fig. (7.2 a).

It can easily be seen that in the range of depths where the asymptotic solution applies, $K_\infty h \geqq 4$, λ_1 and λ_2 remain almost constant. On the other hand, λ_3, which depends on P, exhibits quite a remarkable variation, as shown in Fig. 7.1.

In Fig. 7.1 we show the variation of λ_3 as a function of the depth $K_\infty h$ for four different input data $I_0 = -0.04$, -0.01, 0.04, 0.1. For $I_0 < 0$, λ_3 decreases with decreasing depth, but for $I_0 > 0$ λ_3 increases with decreasing depth up to a "critical depth" (corresponding to $I = 0$) and from there on starts to decrease.

Figure 7.2 shows the group envelopes (dashed line) and wave envelope (solid lines) at a fixed instant for $\varepsilon = 0.2$, at the following four locations: (a) – infinitely deep water, $P_0 = 2, I_0 = 0.1$; (b) $K_\infty h = 11.2$, $P = 2.2$, $I = 0.052$; (c) $K_\infty h = 5.7$, $P = 2.45$, $I = 0.01$; (d) $K_\infty h = 4.2$, $P = 2.75$, $I = -0.028$.

In Fig. 7.2 a we have added a portion of the wavy-surface (thin solid line) as well as the lengths λ_1, λ_2, and λ_3. Note that the supergroups (namely: the exterior and interior group envelopes) are fixed in space, while the wave envelope moves with group-velocity and the waves themselves with the phase velocity. Similar sketches to Fig. 7.2 were obtained for the three other cases given in Fig. 7.1.

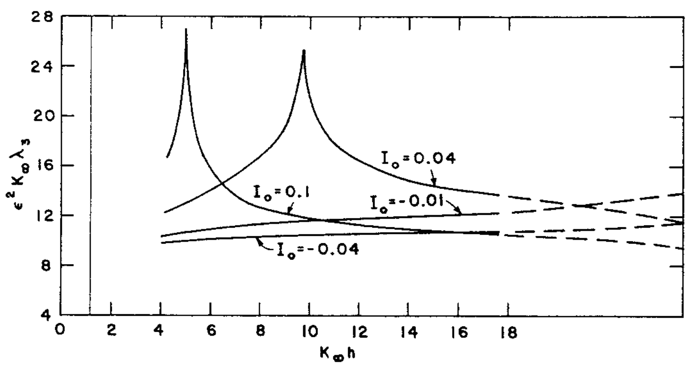

Figure 7.1
The "supergroup" length λ_3 as a function of depth h.

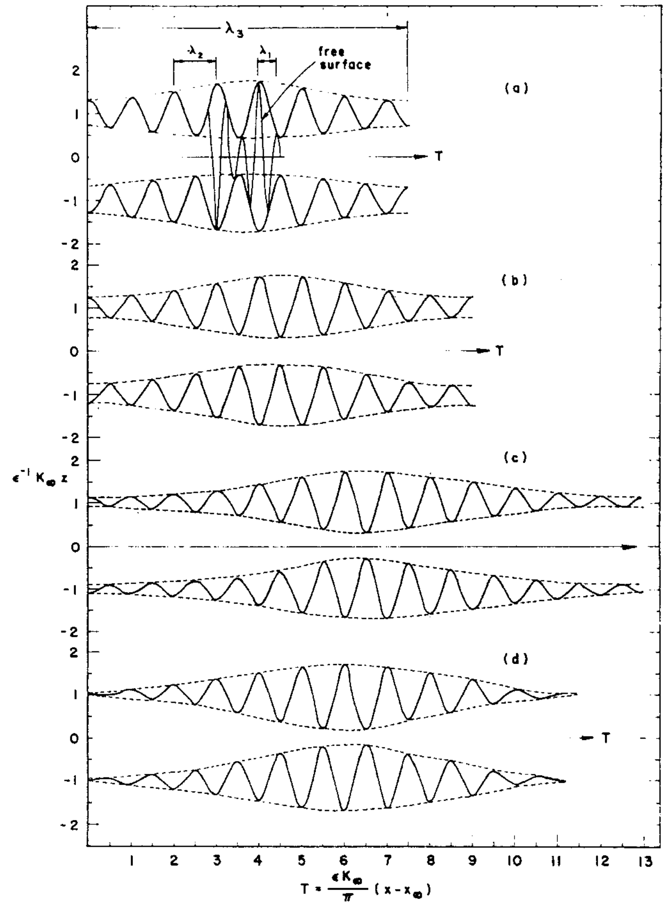

Figure 7.2
The group envelope (- - - -) and wave envelope (———) at t = constant for $P_0 = 2, I_0 = 0.1, \varepsilon = 0.2$ at $K_\infty h$ = (a) ∞, (b) 11.2, (c) 5.7 and (d) 4.2.

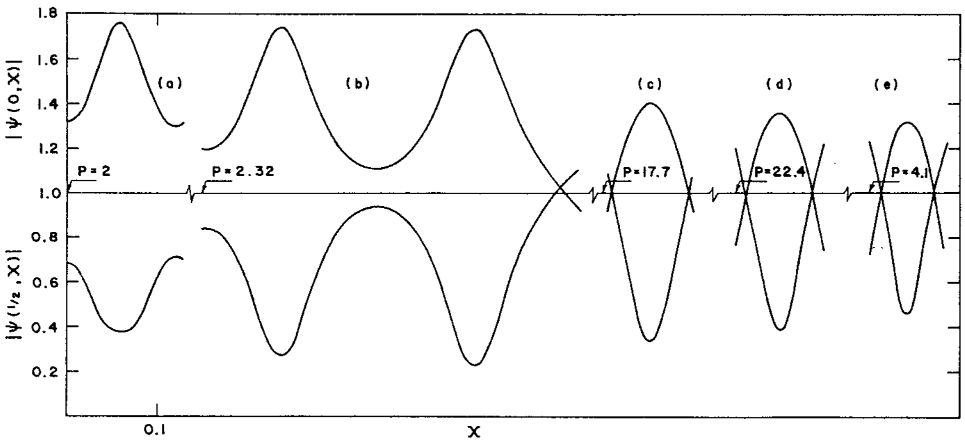

Figure 7.3
The group-envelopes, for $P_0 = 2, I_0 = 0.1$, at $K_\infty h \cong$ (a) ∞, (b) 7.11–4.8, (c) 1.32–1.29, (d) 1.12–1.11, (e) 0.96–0.95.

In order to complete the picture for shallower water depth we present in Fig. 7.3 the group envelopes at five locations: $P =$ a: (2, 2.05), b: (2.32, 2.63), c: (17.7, 22.8), d: (−22.4, −17.5), e: (−4.1, −3.9), as obtained from a numerical solution of the system (4.3 a, b) for the same input data as in Fig. 7.2 assuming $\mu = (2\pi)^2 - 10X$.

The results in Fig. 7.3 indicate that λ_3 continues to shorten and that the intensity of modulation decreases.

8. The mean flow field

We express the mean flow $u = \partial\varphi_0/\partial x = -\partial\tilde{\psi}/\partial z$, $v = \partial\varphi_0/\partial z = \partial\tilde{\psi}/\partial x$ through the stream function $\tilde{\psi}$

$$\tilde{\Psi} = \frac{g^2 K\gamma\varepsilon^{-1}}{4\pi\Omega^3 h}(|D_0|^2 + 2|D_1|^2)Z + \frac{g^3\gamma\varepsilon^{-1}}{4\pi\Omega^3\Omega'}\cdot\frac{\beta_2(D_0^* D_1 + D_0 D_1^*)\, sh(2(Z+H))\cos(2\pi T)}{\frac{g\gamma}{2\pi\varepsilon\Omega'}sh(2H) - ch(2H)} + \frac{g^2\gamma\varepsilon^{-1}}{4\pi\Omega^3\Omega'}\cdot\frac{\beta_2|D_1|^2\, sh(4(Z+H))\cos(4\pi T)}{\frac{g\gamma}{2\pi\varepsilon\Omega'}sh(4H) - 2ch(4H)} + \text{constant} \quad (8.1)$$

where $Z = \pi\varepsilon z/\Omega'\gamma$, $H = \pi\varepsilon h/\Omega'\gamma$ and $\tilde{\Psi} = \varepsilon^{-2}\Omega^3\tilde{\psi}/g^2$ are dimensionless quantities. The constant in Eq. (8.1) is chosen so that $\tilde{\Psi} = 0$ at the bottom. The mean free surface ζ_0 is given by Eq. (2.11)

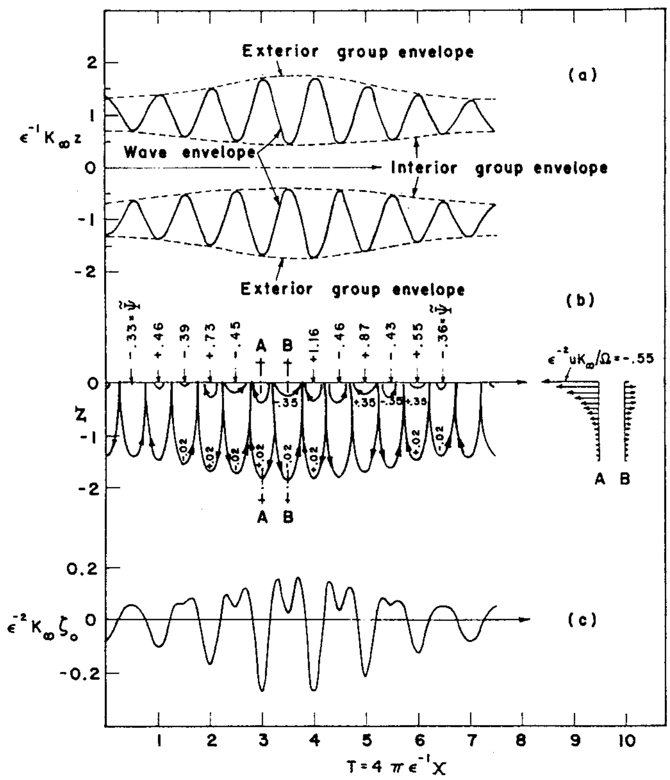

Figure 8.1
The flow field for $P_0 = 2$, $I_0 = 0.1$, $\varepsilon = 0.2$ at $K_\infty h = \infty$. (a) group-envelopes and wave envelope (b) mean flow stream-lines, (c) mean free-surface.

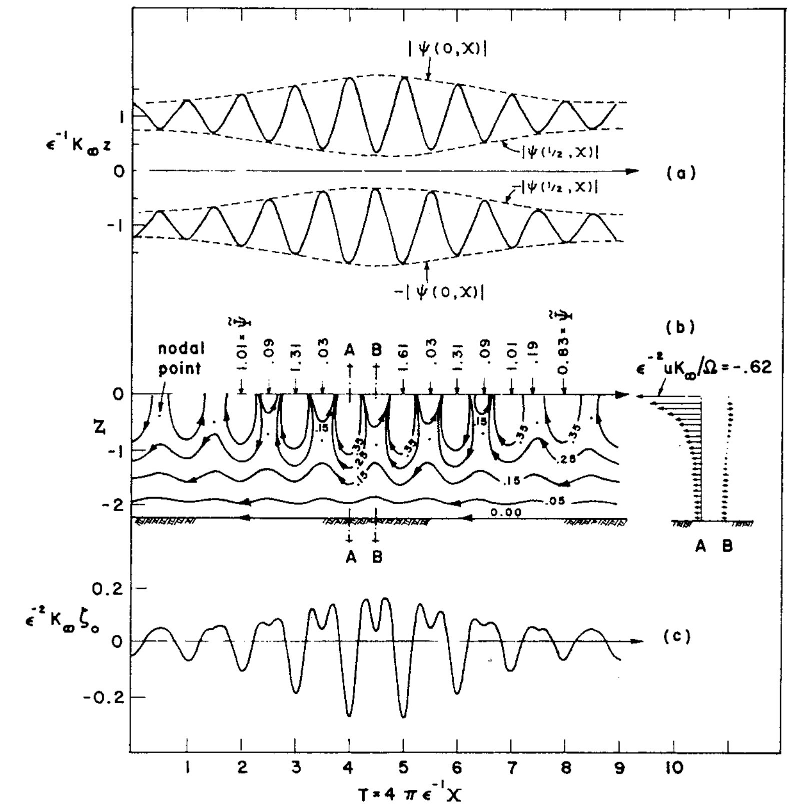

Figure 8.2
As in 8.1 for $K_\infty h = 11.2$.

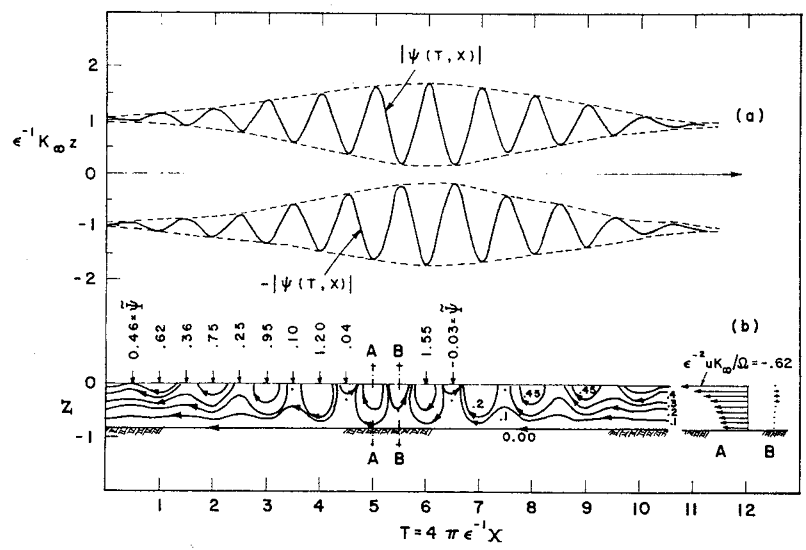

Figure 8.3
As in 8.1 for $K_\infty h$ = 4.2.

The stream-function $\Psi(T, Z)$ as well as the mean free-surface for cases a, b and d of Fig. 7.2 are presented in Figs. 8.1, 8.2 and 8.3 respectively. These figures demonstrate the rather complicated structure of the wave induced mean flow field.

Some of the main features are: (i) the mean current, which is shown in part (b) of the figures, as well as the mean free surface, in part (c) exhibit a somewhat cellular structure influenced by the wave envelope variations; (ii) a dominant adverse current appears underneath the high waves and a much weaker, positive current (in the wave propagation direction) under the low waves, for the shallower cases the positive currents almost disappears; (iii) the magnitude of the maximum adverse currents at the free surface is almost the same for all three depths ($K_\infty h = \infty$, 11.2 and 4.2); (iv) one can notice the tendency of the flow fields to become more uniform in the lower parts and on the sides of the supergroups (where the modulation amplitudes get much smaller); (v) there is a set-down in the mean free surface accompanying the peaks of the wave envelope and a smaller set-up accompanying their troughs.

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Appendix A: Decoupling of the system (2.9) for periodic cases

Assuming A and φ_{10} periodic in τ with period γ, the solution of (2.9) is expanded in a Fourier series:

$$A(\xi, \tau) = \varepsilon \left(\frac{g^3}{2\,\Omega^5\,\Omega'} \right)^{1/2} e^{-i\varepsilon^{-2}\overline{|A|}^2 \int_{\xi_\infty}^{\xi} \frac{\alpha_2}{\Omega'} \mathrm{d}\xi} \left\{ D_0(\xi) + \sum_{n=1}^{\infty} \cdot \left\{ D_n(\xi)\, e^{\frac{2\pi i n \tau}{\gamma}} + D_{-n}(\xi)\, e^{\frac{-2\pi i n \tau}{\gamma}} \right\} \right\}$$

where α_2 is given by (3.2 b) (A.1)

$$\varphi_{10}(\xi, \tau, z) = Q_1(\xi)\,\tau + Q_2(\xi) + \sum_{n=1}^{\infty} a_n(z) \cdot \left\{ b_n(\xi)\, e^{\frac{2\pi i n \tau}{\gamma}} + b_{-n}(\xi)\, e^{\frac{-2\pi i n \tau}{\gamma}} \right\}. \tag{A.2}$$

Substituting (A.2) in (2.9 b) and using the b.c. (2.9 d) yields

$$a_n = \cosh\left[\frac{2\,\pi\,\varepsilon\, n(z+h)}{\Omega'\,\gamma}\right], \qquad n = 1, 2, \ldots \tag{A.3}$$

(same as 3.3 a)

From (A.3) and the b.c. (2.9 c) we obtain

$$b_n(\xi) = \frac{\varepsilon^{-2} g^3 \beta_2}{2\,\Omega} \; \frac{C_n(\xi)}{\frac{2\,\pi\, n\, g}{\varepsilon\,\Omega'\,\gamma} \sinh\left[\frac{2\,\pi\, n\,\varepsilon\, h}{\Omega'\,\gamma}\right] - \left[\frac{2\,\pi\, n}{\gamma}\right]^2 \cosh\left[\frac{2\,\pi\, n\,\varepsilon\, h}{\Omega'\,\gamma}\right]} \tag{A.4}$$

(same as 3.3 b)

where $C_n(\xi)$ are the Fourier coefficients of $(|A|^2)_\tau$:

$$(|A|^2)_\tau = \sum_{n=1}^{\infty} \{C_n(\xi)\, e^{\frac{2\pi i n \tau}{\gamma}} + C_{-n}(\xi)\, e^{\frac{-2\pi i n \tau}{\gamma}}\}; \quad C_n(\xi) \leqq 0(\varepsilon^2). \tag{A.5}$$

(same as 3.4)

Substituting (A.3) and (A.4) in (A.2) and differentiating twice with respect to τ yields:

$$\varphi_{10_{\tau\tau}}|_{z=0} = \sum_{n=1}^{\infty} \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega} \frac{\{C_n(\xi)\, e^{\frac{2\pi i n \tau}{\gamma}} + C_{-n}(\xi)\, e^{\frac{-2\pi i n \tau}{\gamma}}\}}{1 - \dfrac{\gamma g}{2\pi n \varepsilon \Omega'}\, th\left[\dfrac{2\pi n \varepsilon h}{\Omega' \alpha}\right]}. \tag{A.6}$$

Thus from (A.5) and (A.6) follows the relation

$$\varphi_{10_{\tau\tau}}|_{z=0} = \Gamma_1 (|A|^2)_\tau + \sum_{n=2}^{\infty} (\Gamma_n - \Gamma_1) \{C_n(\xi)\, e^{\frac{2\pi i n \tau}{\gamma}} + C_{-n}\, e^{\frac{-2\pi i n \tau}{\gamma}}\} \tag{A.7}$$

where

$$\Gamma_n = \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega\left(1 - \dfrac{\gamma g}{2\pi n \varepsilon \Omega'}\, th\left[\dfrac{2\pi n \varepsilon h}{\Omega' \gamma}\right]\right)}, \quad n = 1, 2, 3, \ldots. \tag{A.8}$$

It can be shown that $\Gamma_n - \Gamma_1 \leqq 0(\varepsilon^{-2})$ for all n. On the other hand, $C_n(\xi)$ are Fourier coefficients. Thus there exists an N, such that

$$\sum_{n=N}^{\infty} \{C_n(\xi)\, e^{\frac{2\pi i n \tau}{\gamma}} + C_{-n}(\xi)\, e^{\frac{-2\pi i n \tau}{\gamma}}\} \leqq \varepsilon^3 \tag{A.9}$$

and then

$$\sum_{n=N}^{\infty} (\Gamma_n - \Gamma_1) \{C_n\, e^{\frac{2\pi i n \tau}{\gamma}} + C_{-n}\, e^{\frac{-2\pi i n \tau}{\gamma}}\} \leqq 0(\varepsilon). \tag{A.10}$$

For $n \leqq N$:

$$\Gamma_n - \Gamma_1 \leqq 0(\varepsilon^{-1}). \tag{A.11}$$

From (A.7), (A.10) and (A.11) we finally obtain the relation

$$\varphi_{10_{\tau\tau}}|_{z=0} = \Gamma_1 |A|^2_\tau + 0(\varepsilon). \tag{A.12}$$

From (A.8) one can show that

$$\Gamma_1 = \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega\left(1 - \dfrac{gh}{(\Omega')^2}\right)} + 0(\varepsilon^{-1}) \tag{A.13}$$

and thus

$$\varphi_{10_{\tau\tau}}|_{z=0} = \Gamma |A|^2_\tau + 0(\varepsilon) \tag{A.14}$$

where

$$\Gamma = \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega\left(1 - \dfrac{g h}{(\Omega')^2}\right)}. \tag{A.15}$$

Equations (A.14) and (A.15) are identical to those obtained for the case of shallow water (see Djordjevic et al. 1978, Stiassnie 1983). Integrating (A.14) with respect to τ yields

$$\varphi_{10_\tau}|_{z=0} = \Gamma\,|A|^2 + Q(\xi) + 0(\varepsilon). \tag{A.16}$$

In order to find Q we introduce a lateral boundary condition of zero averaged (over τ) mass flow which is appropriate for an impervious beach, as follows:

$$h\bar{U} = \frac{K g}{2\Omega}\overline{|A|^2} = 0 \tag{A.17}$$

where $U = \varepsilon\,\varphi_{10_x}$ is the wave induced mean current velocity, and the bar indicates averaging.

From Eqs. (A.16), (A.15) and (A.17) we obtain:

$$Q(\xi) = -\varepsilon^{-2}\overline{|A|^2}\left\{\frac{g K \Omega'}{2\Omega h} + \frac{g^2\beta_2}{2\Omega\left[1 - \dfrac{g h}{(\Omega')^2}\right]}\right\}. \tag{A.18}$$

Integrating of Eq. (A.16) with respect to τ yields

$$\varphi_{10} = \frac{\varepsilon^{-2} g^2 \beta_2}{2\Omega\left[1 - \dfrac{g h}{(\Omega')^2}\right]}\int |A|^2\,d\tau + Q(\xi)\,\tau + \varepsilon\,Q_2(\xi). \tag{A.19}$$

Substituting Eq. (A.19) into (2.7d), and suppressing secular terms in t (which grow monotonically and boundlessly) we obtain:

$$\varphi_{20}(\xi) = \frac{-\varepsilon^{-2} g K \Omega' \overline{|A|^2}}{2\Omega h}. \tag{A.20}$$

(same as 3.5)

Averaging the two expressions for φ_{0_τ} in (A.2) and in (A.16) and comparing the results, yields

$$Q_1(\xi) = -\varepsilon^{-2}\overline{|A|^2}\frac{g K \Omega'}{2\Omega h}. \tag{A.21}$$

Finally, substitution of (A.16), (A.18) and (A.20) into (2.9a) gives Eq. (3.1). Substituting (A.21) in (A.2) gives (3.3).

Appendix B: Solution of Eqs. (4.3 a, b) for constant depth

Input data: $J_1, J_3, P;\quad I = J_3 - J_1^2$

$$C_1 = \frac{1}{7}(4J_1 - P)\left\{1 - \left[1 - \frac{14I}{(4J_1 - P)^2}\right]^{1/2}\right\} \tag{B.1}$$

$$C_2 = P \cdot \left[1 - \left(1 + \frac{2I}{P^2}\right)^{1/2}\right] \tag{B.2}$$

$$c = \max(C_1, C_2);\quad d = \min(C_1, C_2) \tag{B.3 a, b}$$

$$a = 2P;\quad b = \frac{2}{7}(4 - P) \tag{B.3 c, d}$$

$$k^2 = 1 - \frac{a-b}{a} \cdot \frac{c-d}{b} \tag{B.4}$$

$$y = -\frac{2\pi^2}{P}\sqrt{7ab} \cdot X \tag{B.5}$$

$cd(y, k)$ is a Jacobian elliptic function of the argument y with modulus k.

$$\tilde{z} = \frac{a \cdot b \cdot (1 - cd^2)}{a - b \cdot cd^2} \tag{B.6}$$

$$|D_1| = \sqrt{\tilde{z}/2};\quad |D_0| = \sqrt{J_1 - \tilde{z}} \tag{B.7 a, b}$$

$$\cos[2(\arg D_1 - \arg D_0)] = \frac{I - 1.5\tilde{z}^2 + (P - 2J_1) \cdot \tilde{z}}{2\tilde{z}(J_1 - \tilde{z})}. \tag{B.7 c}$$

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Abstract

An approximate analytical solution describing the shoaling of modulated wave-trains is presented. This solution provides new information about the wave field evolution as well as about the wave-induced mean current and set down.

Zusammenfassung

Eine angenäherte analytische Lösung wird gegeben, die das Brechen von modulierten Wellenzügen beschreibt. Die Lösung gibt neue Informationen über die Entwicklung des Wellenfeldes wie auch über die mittlere Strömung und mittlere Höhenänderung, die von den Wellen induziert wird.

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